LOCALIZING SMOKE AND GAS SOURCES USING PHYSICS-INSPIRED SPARSE BAYESIAN LEARNING

DLR-KN

Dmitriy Shutin

DLR – German Aerospace Research

10,685 employees across 55 institutes and facilities at 30 sites

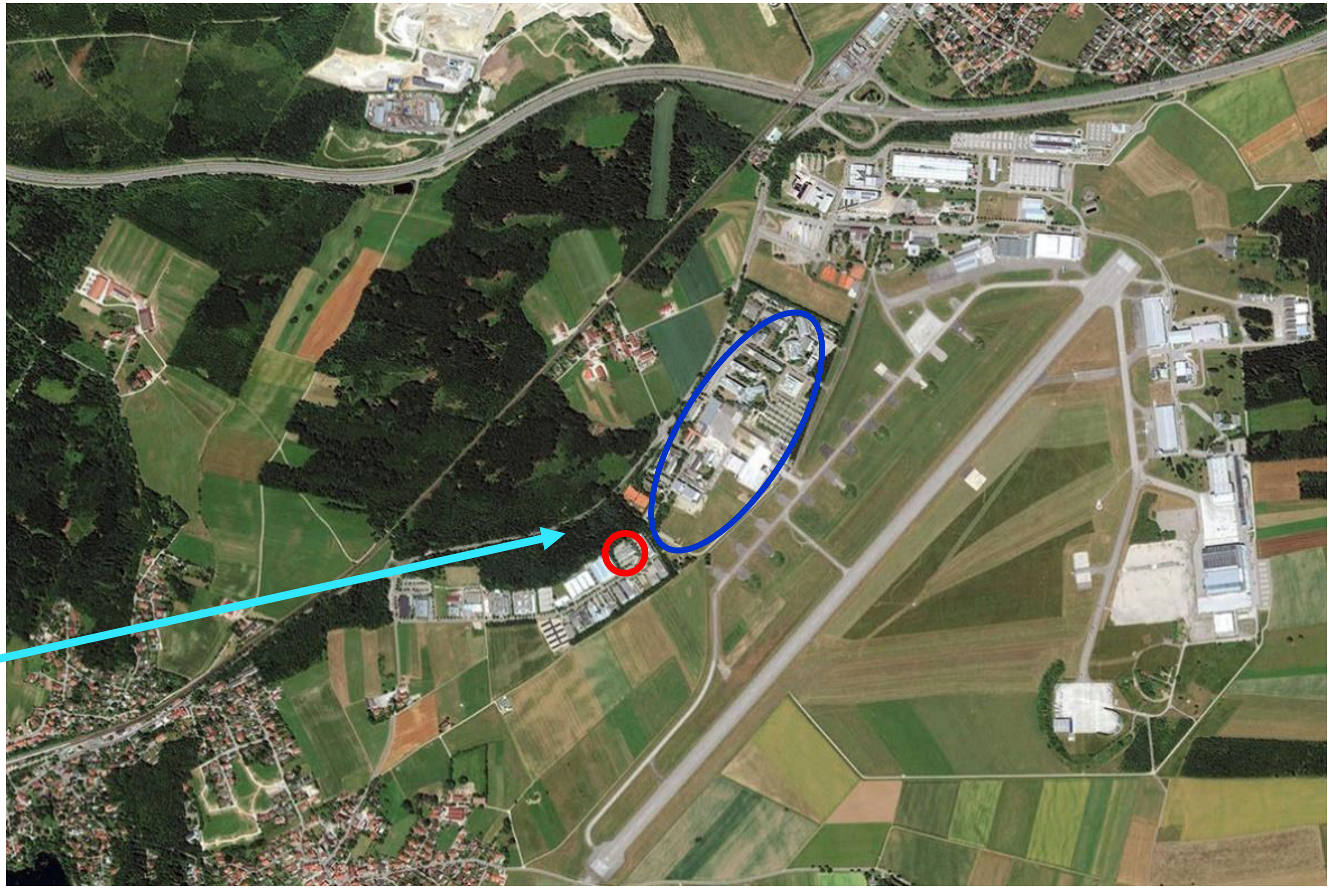
Head of Board: Prof. Dr.-Ing. Anke Kaysser-Pyzalla

Programmatic lines: space, aeronautics, transportation and energy as well as security and digitalisation

Offices in Brussels, Paris, Tokyo and Washington

Annual budget of about 1370 Mio. €

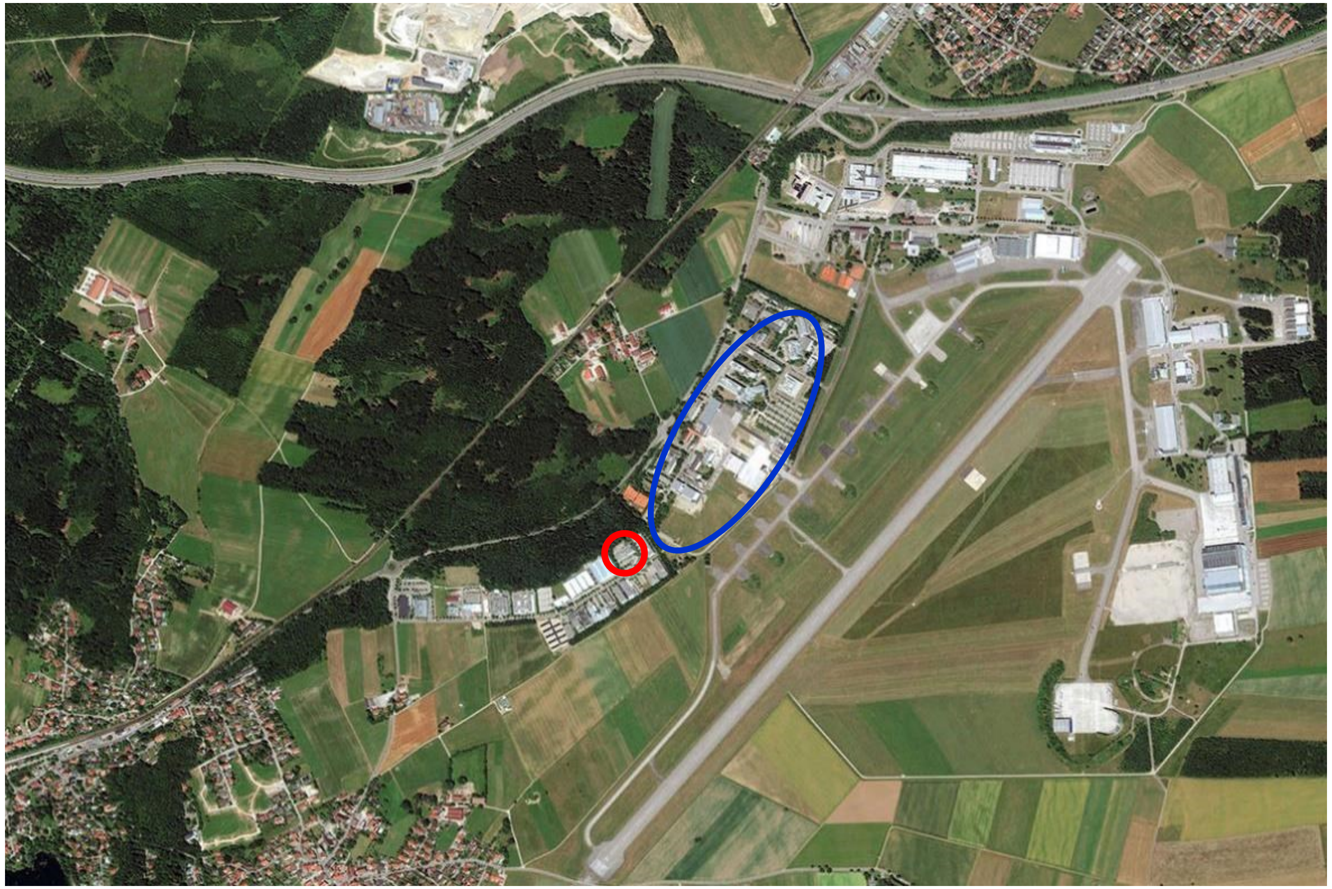
DLR – German Aerospace Research



DLR Research Site Oberpfaffenhofen

- Institutes:
  - **Communications and Navigation (KN)**
  - Microwaves and Radar (HR)
  - Remote Sensing Technology (MF)
  - Atmospheric Physics (PA)
  - Robotics and Mechatronics (RM)
  - System Dynamics and Control (SR)

- Scientific-Technical Facilities:
  - German Remote Sensing Data Center (DFD)
  - Flight Experiments (FX)
  - Space Operations and Astronaut Training (RB)
  - Robotics and Mechatronics Center (RMC)
  - Galileo Control Center (GCC)
  - Galileo Competence Center (GK)



Institute of Communications and Navigation

230 employees

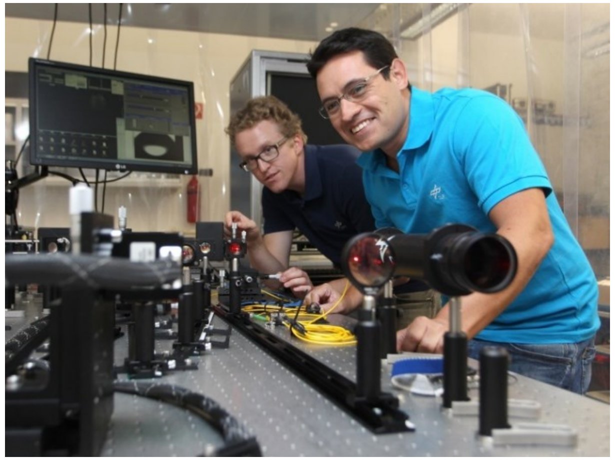
at the DLR sites

Oberpfaffenhofen, Neustrelitz, and Aachen

The institute is engaged in the design, analysis and realization of systems for communication and navigation in the fields of space, aeronautics, land and ship transportation and security.

The work ranges from the scientific fundaments to technology demonstration in a real environment and technology transfer in cooperation with industry.

The Institute's work is oriented toward four missions that have a direct benefit for society and the economy.



Images: Enno Kapitza

Our Missions

GLOBAL CONNECTIVITY

- System Concepts for VHTS and Mega-Constellations
- Data Repatriation from Space
- New Communication Standards for Aviation and Maritime Traffic

GLOBAL POSITIONING

- Kepler System Architecture and Key Technologies
- System Monitoring and Threat Analysis
- Alternative PNT Systems for aviation and Maritime Transport

AUTONOMY AND COOPERATION

- Robust Communication and Reliable Positioning
- Cooperative Systems and Traffic Assistance Systems
- Swarm Systems for Exploration

CYBERSECUIRITY

- Cryptographic Algorithms and Quantum Key Distribution
- Security Measures for Signals and Sensors
- Architectures and Technologies for Secure Systems and Infrastructures

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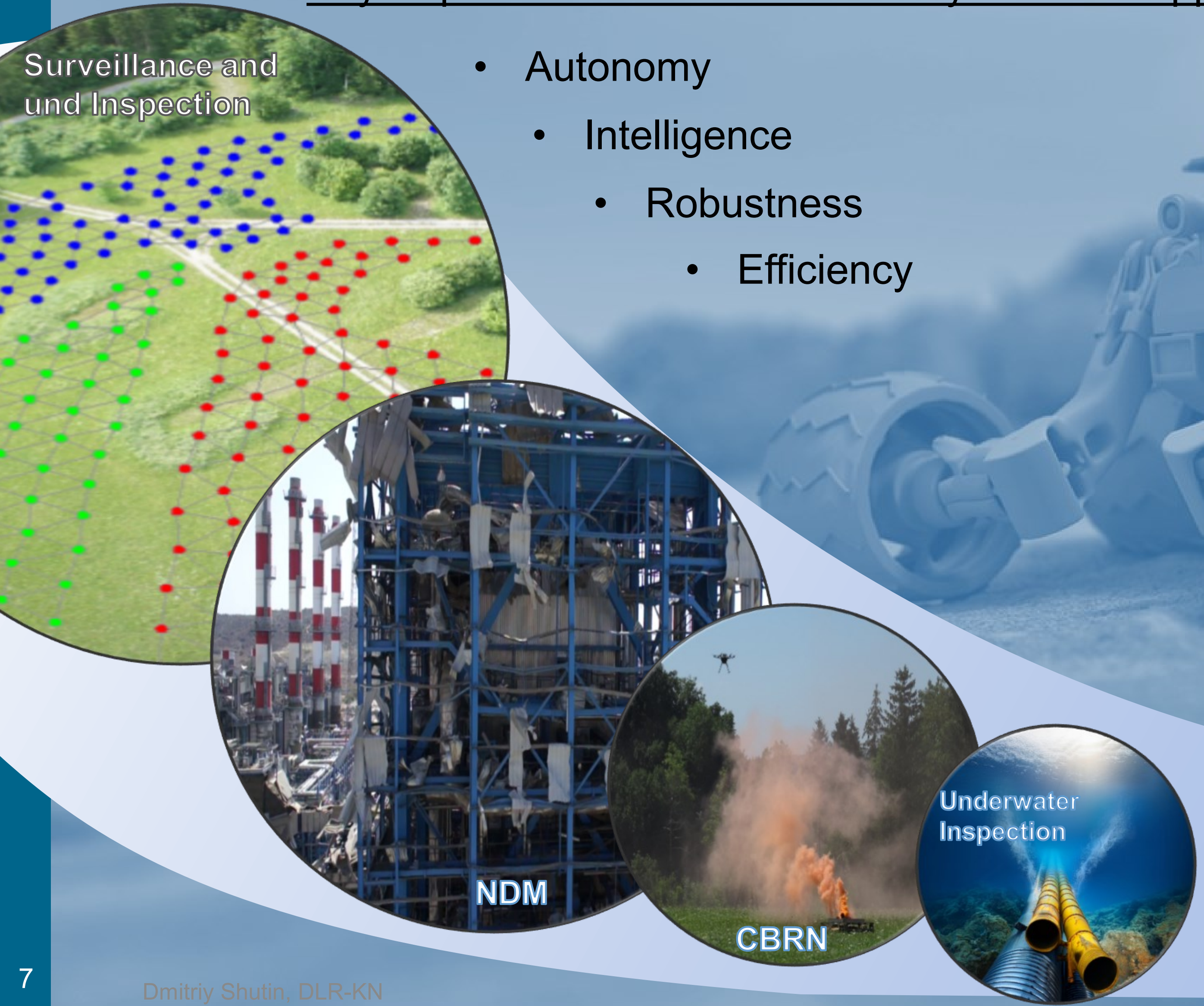
Swarms – towards cooperative navigation and exploration

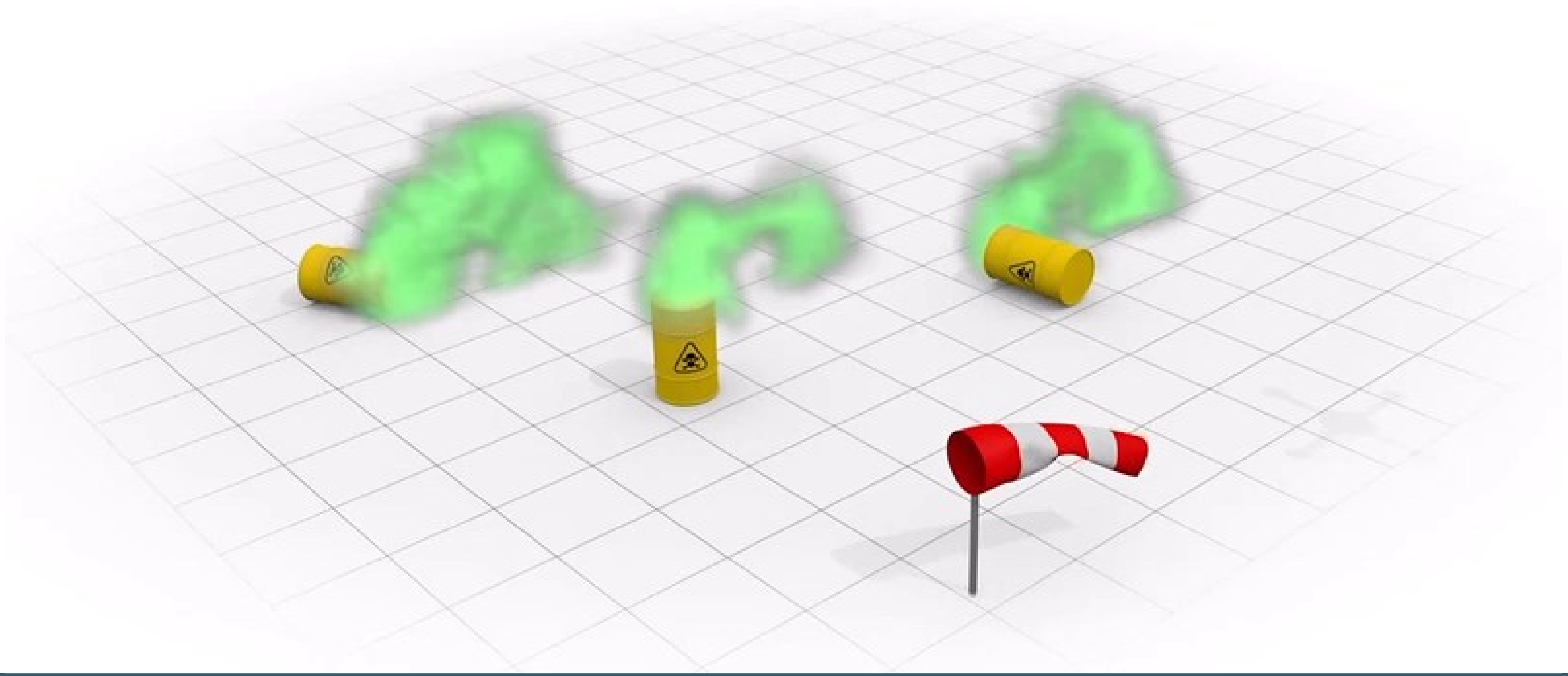
Key requirements for future robot systems for applications in exploration, NDM and security

- Autonomy
  - Intelligence
    - Robustness
      - Efficiency



Swarms can address all these challages!

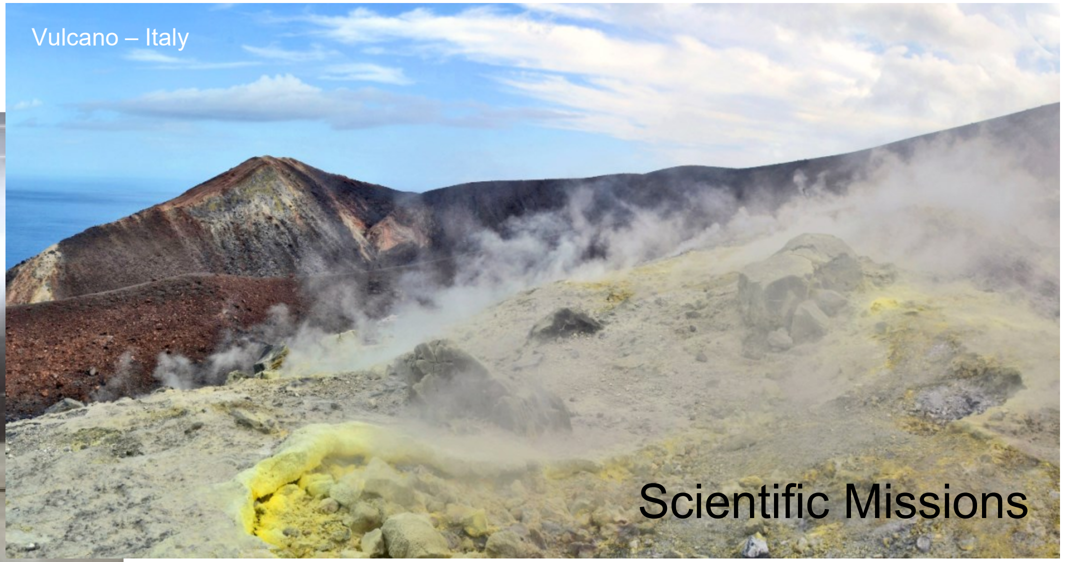
… in planetary mission, NDM, as well as in a number of security tasks,
…or air, water and under!



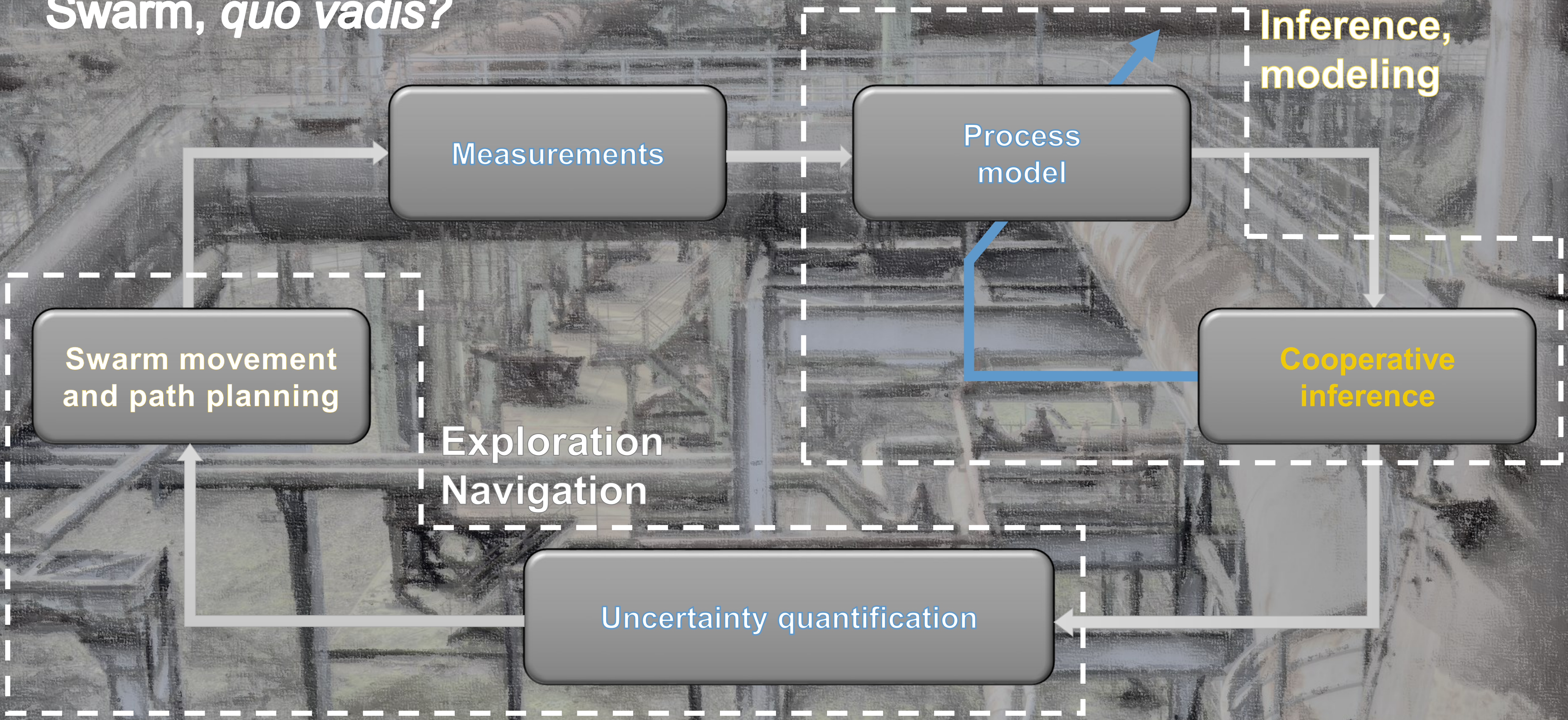
SWARMS FOR GAS SOURCE LOCALIZATION

Key application areas

Gas Distribution Mapping with Mobile Robots



Swarm, *quo vadis?*



PROBLEM MODELING

Physical model of gas propagation
Advection-Diffusion

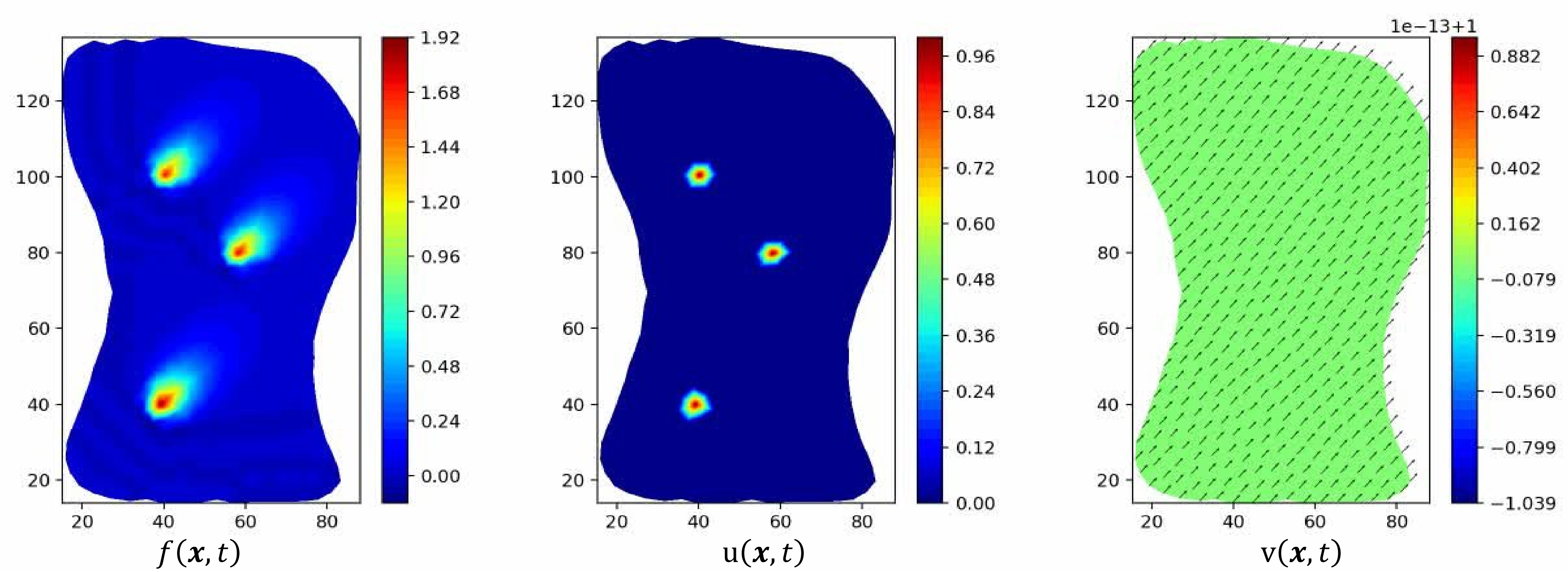
Process model

$$\frac{\partial f(\boldsymbol{x},t)}{\partial t} - \kappa \nabla^2 f(\boldsymbol{x},t) + \boldsymbol{v}(\boldsymbol{x},t) \nabla f(\boldsymbol{x},t) \quad = u(\boldsymbol{x},t)$$

Measurement model:

$$y_f(\boldsymbol{x},t) = f(\boldsymbol{x},t) + \varepsilon_f$$

$$\boldsymbol{w}_{\boldsymbol{v}}(\boldsymbol{x},t) = \boldsymbol{v}(\boldsymbol{x},t) + \boldsymbol{\varepsilon}_{\boldsymbol{v}}$$



$f(\boldsymbol{x},t)$
Concentration

$u(\boldsymbol{x},t)$
Source distribution

$v(\boldsymbol{x},t)$
Advection (wind)

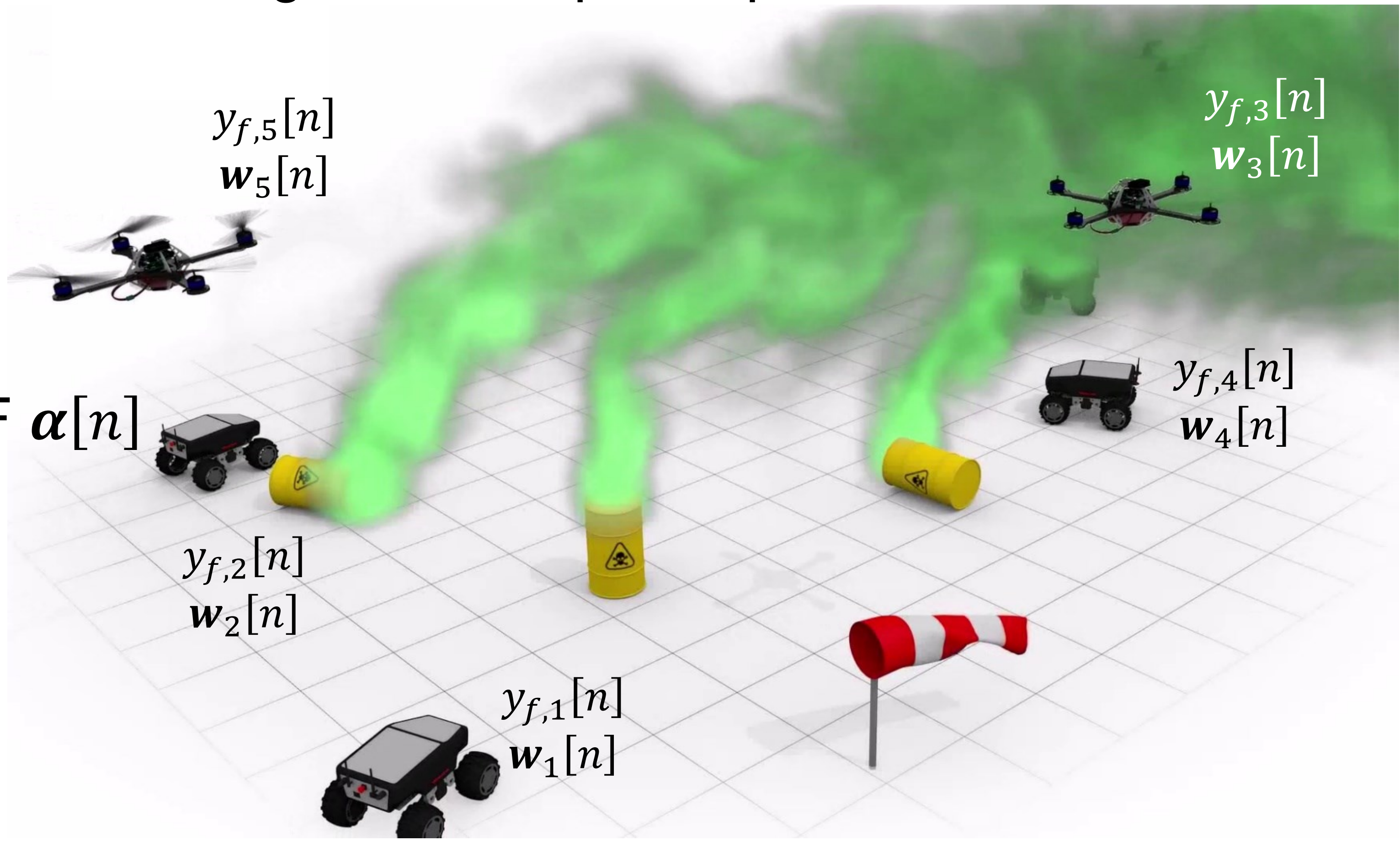
Measurement model for K agents

- Concentration $\boldsymbol{f}[n]$ measurement with gas sensor point spread-function (PSF) $\boldsymbol{m}_k[n]$

$$y_{f,k}[n] = \boldsymbol{m}_k[n]^T \boldsymbol{f}[n] + \varepsilon_{f,k}\ [n]$$

- Wind measurement with PSF $\boldsymbol{\alpha}[n]$

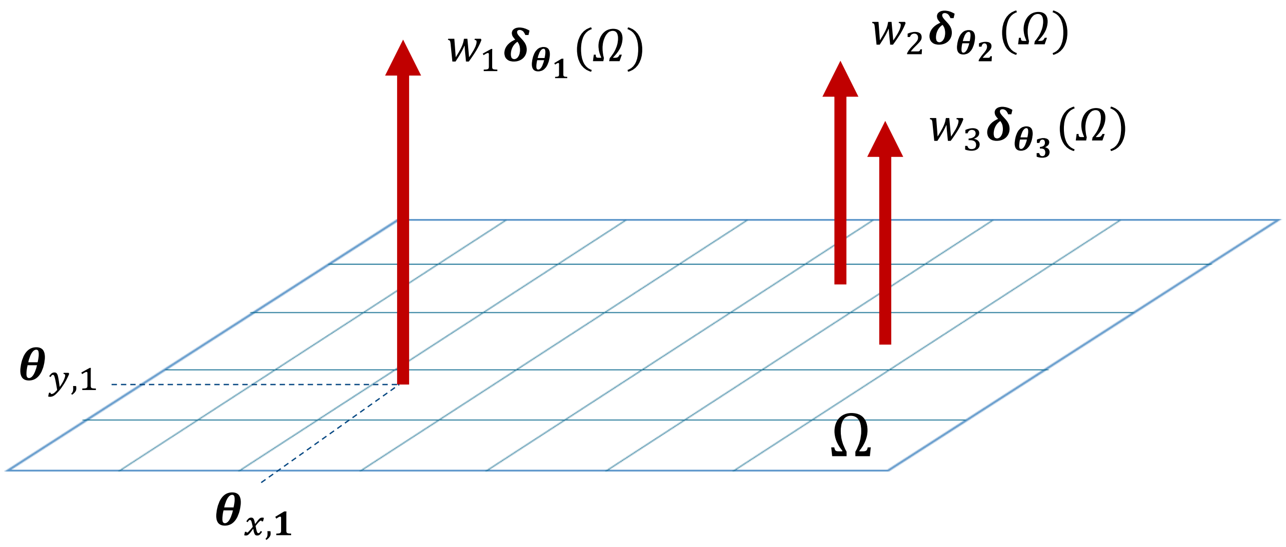
$$\boldsymbol{w}_k[n] = \begin{bmatrix} \boldsymbol{v}_{\mathrm{x}}[n]^T \\ \boldsymbol{v}_{y}[n]^T \end{bmatrix} \boldsymbol{\alpha}_k[n] + \boldsymbol{\varepsilon}_{v,k}\ [n]$$



How to model gas sources
Stationary source assumption

- Source modeling:

$$u(\boldsymbol{x},t) \equiv u(\boldsymbol{x}) = \sum_{l=1}^{L_{\text{true}}} w_l \, \delta_{\boldsymbol{\theta}_l}(\Omega)$$

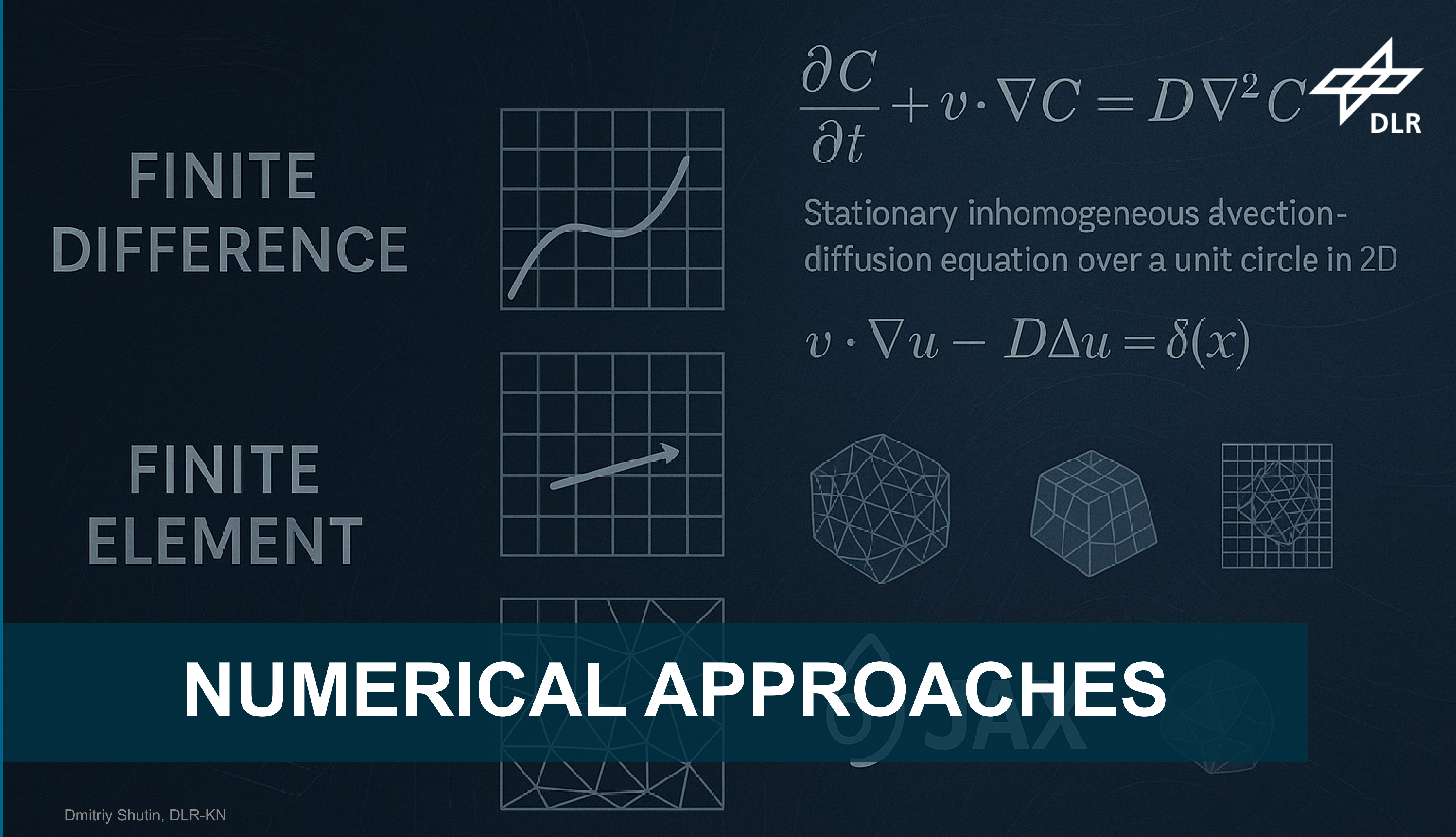


- Sparse Bayesian Learning:

Surrogate model

$$u(\boldsymbol{x}) \approx \sum_{l=1}^{L} w_l \, \delta_{\boldsymbol{\theta}_l}(\Omega)$$

- Overfit assumption: $L \gg L_{\text{true}}$
- w_l are **sparse**, i.e. some source weight are zero $w_l = 0$



NUMERICAL APPROACHES

Discretization: from continuum to finite dimensions
Numerical approaches (PDE in Weak form)

- Weak form of a PDE: project the equation on some (test) function $\varphi(\boldsymbol{x}, t)$

$$\left\langle \varphi(\boldsymbol{x}, t), \frac{\partial f(\boldsymbol{x}, t)}{\partial t} \right\rangle - \langle \varphi(\boldsymbol{x}, t), \kappa \nabla^2 f(\boldsymbol{x}, t) \rangle + \langle \varphi(\boldsymbol{x}, t), \boldsymbol{v}(\boldsymbol{x}, t) \nabla f(\boldsymbol{x}, t) \rangle = \langle \varphi(\boldsymbol{x}, t), u(\boldsymbol{x}, t) \rangle$$

- Galerkin method: *C*-dimensional approximation with fixed (spatial) basis functions $\psi_i(\boldsymbol{x}), i = 1 \ldots C$

$$f(\boldsymbol{x}, t) \approx \sum_{i=1}^{C} f_i(t) \psi_i(\boldsymbol{x})$$

Concertation distribution

$$u(\boldsymbol{x}, t) \approx \sum_{i=1}^{C} u_i(t) \psi_i(\boldsymbol{x})$$

Source signal

$$\varphi(\boldsymbol{x}, t) \approx \sum_{i=1}^{C} \varphi_i(t) \psi_i(\boldsymbol{x})$$

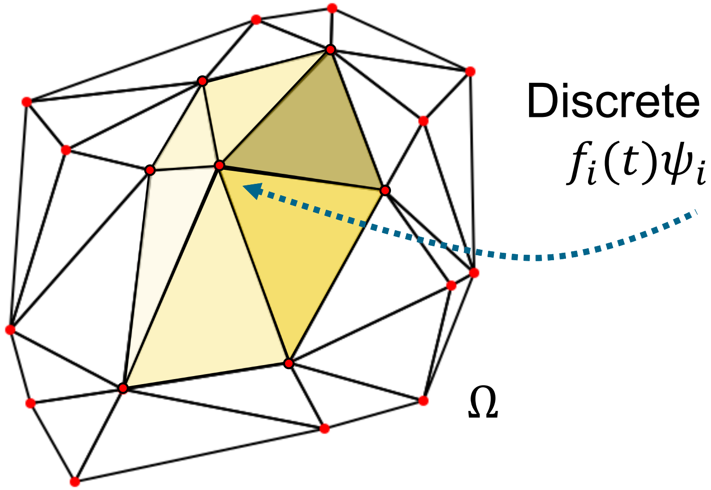
Test function

$$v(\boldsymbol{x}, t) \approx \begin{bmatrix} \sum_{i=1}^{C} v_{x,i}(t) \psi_i(\boldsymbol{x}) \\ \sum_{i=1}^{C} v_{y,i}(t) \psi_i(\boldsymbol{x}) \end{bmatrix}$$

Wind velocity

Discretization: from continuum to finite dimensions, cont'd

- Basis functions $\psi_i(\boldsymbol{x}), i = 1 \dots C$ (finite elements)
  - Defined on some discretized exploration domain Ω
  - Delaunay triangulation is often used



- PDE is parameterized with finite-dimensional parameter vectors

$$\boldsymbol{f}(t) = \begin{bmatrix} f_1(t) \\ \vdots \\ f_C(t) \end{bmatrix} \qquad \mathbf{u}(t) = \begin{bmatrix} u_1(t) \\ \vdots \\ u_C(t) \end{bmatrix} \qquad \boldsymbol{\varphi}(t) = \begin{bmatrix} \varphi_1(t) \\ \vdots \\ \varphi_C(t) \end{bmatrix} \qquad \boldsymbol{v}_{\mathrm{x}}(t) = \begin{bmatrix} v_{\mathrm{x},1}(t) \\ \vdots \\ v_{\mathrm{x},C}(t) \end{bmatrix} \qquad \boldsymbol{v}_{\mathrm{y}}(t) = \begin{bmatrix} v_{\mathrm{y},1}(t) \\ \vdots \\ v_{\mathrm{y},C}(t) \end{bmatrix}$$

Discretization: from continuum to finite dimensions, cont'd

- Source term

$$\langle \varphi(\boldsymbol{x},t), u(\boldsymbol{x},t)\rangle \approx \int_{\Omega} \sum_{i=1}^{C} u_i\,(t)\,\psi_i\,(\boldsymbol{x}) \sum_{j=1}^{C} \varphi_j\,(t)\,\psi_j\,(\boldsymbol{x})d\boldsymbol{x} = \boldsymbol{\varphi}(t)^{\boldsymbol{T}}\boldsymbol{A}\,\boldsymbol{u}(t), \quad [\boldsymbol{A}]_{i,j} = \int_{\Omega} \psi_i\,(\boldsymbol{x})\psi_j\,(\boldsymbol{x})d\boldsymbol{x}$$

- Time-derivative term

$$\left\langle \varphi(\boldsymbol{x},t), \frac{\partial f(\boldsymbol{x},t)}{\partial t} \right\rangle \approx \boldsymbol{\varphi}(t)^{\boldsymbol{T}}\boldsymbol{A}\frac{d\boldsymbol{f}(t)}{dt}$$

- Diffusion term

$$\langle \varphi(\boldsymbol{x},t), \kappa\nabla^2 f(\boldsymbol{x},t)\ \rangle \approx \kappa\ \boldsymbol{\varphi}(t)^{\boldsymbol{T}}\boldsymbol{D}\ \boldsymbol{f}(t)$$

- Advection term

$$\langle \varphi(\boldsymbol{x},t), \boldsymbol{v}(\boldsymbol{x},t)\nabla f(\boldsymbol{x},t)\rangle \approx\ \boldsymbol{\varphi}(t)^{\boldsymbol{T}}[\boldsymbol{v}_{\mathrm{x}}(t) \circ \boldsymbol{G}_{\boldsymbol{x}}\boldsymbol{f}(t)] + \boldsymbol{\varphi}(t)^{\boldsymbol{T}}\left[\boldsymbol{v}_{\mathrm{y}}(t) \circ \boldsymbol{G}_{\boldsymbol{y}}\boldsymbol{f}(t)\right]$$

Discretization: from continuum to finite dimensions, cont'd
Discretization in time

- Discrete-space, continuous-time equation

$$\boldsymbol{A}\frac{d\boldsymbol{f}(t)}{dt} - \kappa \boldsymbol{D}\boldsymbol{f}(t) + \boldsymbol{v}_{\mathrm{x}}(t) \circ \boldsymbol{G}_{\boldsymbol{x}}\boldsymbol{f}(t) + \boldsymbol{v}_{\boldsymbol{y}}(t) \circ \boldsymbol{G}_{y}\boldsymbol{f}(t) \quad = \boldsymbol{A}\,\boldsymbol{u}(t)$$

- Discretization in time: $t = n \triangle_T,\ n = 0,1,2 \ldots$

$$\frac{\boldsymbol{A}}{\triangle_T}\left(\boldsymbol{f}[n] - \boldsymbol{f}[n-1]\right) - \kappa \boldsymbol{D}\boldsymbol{f}[n] + \boldsymbol{v}_{\mathrm{x}}[n] \circ \boldsymbol{G}_{\boldsymbol{x}}\boldsymbol{f}[n] + \boldsymbol{v}_{\boldsymbol{y}}[n] \circ \boldsymbol{G}_{y}\boldsymbol{f}[n] = \boldsymbol{A}\,\boldsymbol{u}[n]$$

- Final step - boundary conditions : $\boldsymbol{B}\boldsymbol{f}[n] = \boldsymbol{b}$

Probabilistic modeling of PDE

Relaxed Gas Model:

$$\frac{1}{\triangle_T}\boldsymbol{A}(\boldsymbol{f}[n]-\boldsymbol{f}[n-1])-\kappa\,\boldsymbol{D}\,\boldsymbol{f}[n]+\boldsymbol{v}_1[n]\circ\boldsymbol{G}_x\,\boldsymbol{f}[n]+\boldsymbol{v}_2[n]\circ\boldsymbol{G}_y\,\boldsymbol{f}[n]-\boldsymbol{A}\boldsymbol{u}[n]=\boldsymbol{r}[n] \neq 0$$

residual ***r[n]*** is zero-mean normal with a precision τ_s

$$p(\text{concentration}\mid\text{wind},\text{sources})\propto e^{-\frac{\tau_s}{2}\boldsymbol{r}[\boldsymbol{n}]^T\boldsymbol{r}[\boldsymbol{n}]}$$

Measurement Model:

$$y_l[n]=\boldsymbol{m}_l[n]^T\boldsymbol{f}[n]+\varepsilon_l[n],\ \ \boldsymbol{w}_l[n]=\begin{bmatrix}\boldsymbol{v}_{\mathrm{x}}[n]^T\\ \boldsymbol{v}_y[n]^T\end{bmatrix}\boldsymbol{\alpha}_l[n]+\boldsymbol{\varepsilon}_{v,l}[n]$$

$$p(\text{measurement}\mid\text{wind},\text{concentration})\propto e^{-\frac{\tau_m}{2}\left(y_l[n]-\boldsymbol{m}_l[n]^T\boldsymbol{f}[n]\right)^2-\frac{\tau_w}{2}\left\|\boldsymbol{w}_l[n]-\begin{bmatrix}\boldsymbol{v}_{\mathrm{x}}[n]^T\\ \boldsymbol{v}_y[n]^T\end{bmatrix}\boldsymbol{\alpha}_l[n]\right\|^2}$$

Bayesian Inference approach: find posterior $p(\text{sources},\text{concentration},\text{wind}\mid\text{measurements})\propto$

$$\underbrace{p(\text{measurement}\mid\text{wind},\text{concentration})}_{\textit{state likelihood}}\ \underbrace{p(\text{concentration}\mid\text{wind},\text{sources})}_{\textit{relaxed model}}\ \underbrace{p(\text{wind})}_{\textit{wind prior ?}}\ \underbrace{p(\text{source})}_{\textit{source prior ?}}$$

Wind prior modeling

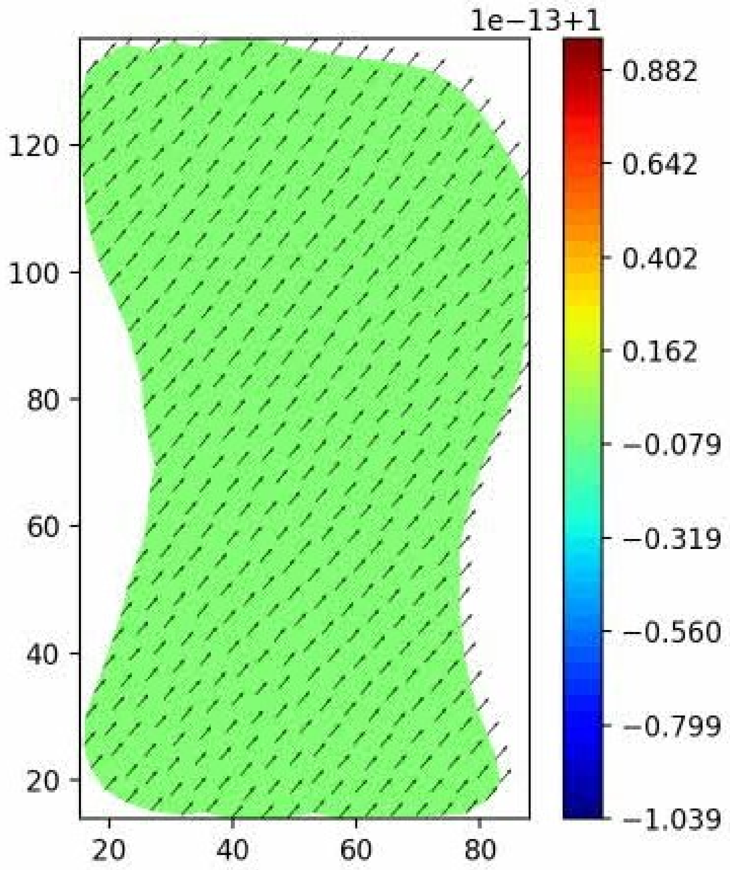
- We do not distinguish between laminar and turbulent components
- X-Y wind directions are assumed statistically independent in space

$$p(\text{wind}) = p(v_x[n])p\big(v_y[n]\big)$$

$$v_x[n] \sim \mathcal{N}\ (\mu_x, \sigma_x^2)$$

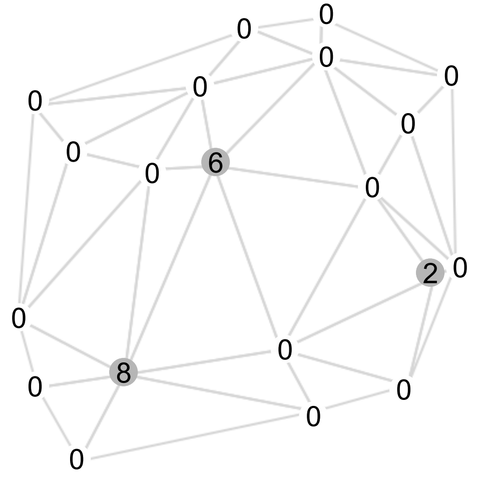
$$v_y[n] \sim \mathcal{N}\ (\mu_x, \sigma_y^2)$$

- Parameters μ_x, σ_x^2 and μ_y, σ_y^2 have to selected (e.g., weather forecast)



Source prior modeling: Sparse Bayesian Learning

- Main assumption: There are a few distinct gas sources, i.e., gas source signal $\boldsymbol{u}[n]$ is sparse, $\forall n \in \mathbb{N}$



$$\mathbf{u}[n] = \begin{bmatrix} 0 \\ 6 \\ 0 \\ \vdots \\ 0 \\ 2 \\ 0 \\ \vdots \\ 8 \\ 0 \\ 0 \end{bmatrix}$$

- Our goal: find $\boldsymbol{u}[n]$ with minimal number of non-zero elements

$$p(\text{source}) \equiv p(\text{source}, \text{hyperparameters}) = p(\text{source} \,|\text{hyperparameters})p(\text{hyperparameters})$$

- Solution: impose sparsity constraints on $p(\text{source})$ with sparse Bayesian Learning

Sparse Bayesian Learning for modeling sources

- Hierarchical Prior: $p(\text{source}, \text{hyperparameters}) = p(\text{source} \mid \text{hyperparameters})\, p(\text{hyperparameters})$

$$p(\text{source} \mid \text{hyperparameters}) = p(u_i[n] \mid \gamma_i) = \mathcal{N}\left(u_i[n] \mid 0, \frac{1}{\gamma_i}\right)$$

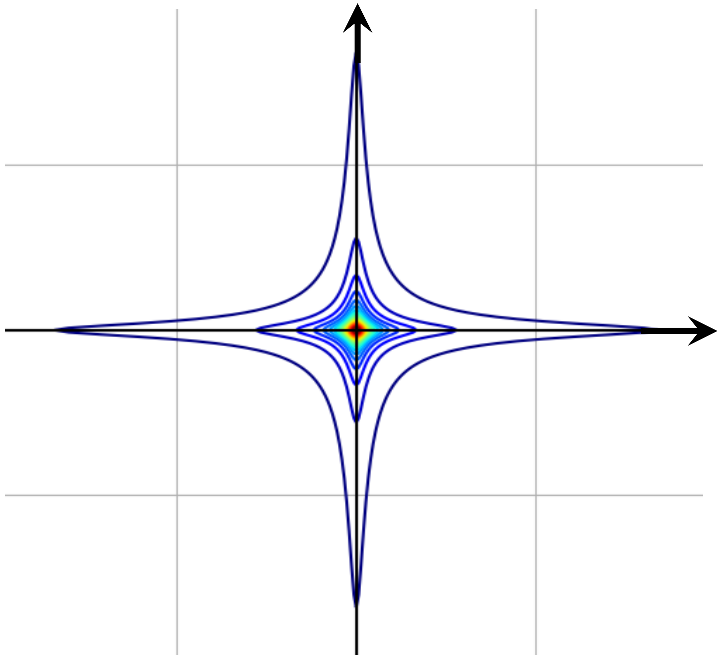
$$p(\text{hyperparameters}) = p(\gamma_i) = Gamma(\gamma_i \mid a, b)$$

- Equivalent (marginalized) source prior

$$p(\text{sources}) = p(u_i[n])$$

$$= \int p(u_i[n] \mid \gamma_i) \cdot p(\gamma_i) d\gamma_i = \text{Student's t PDF}$$

- SBL requires estimation of hyperparameters $\gamma_i, i = 1, \dots, C$

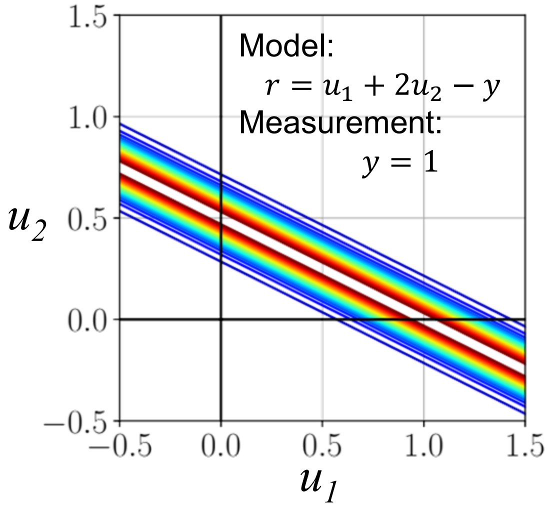


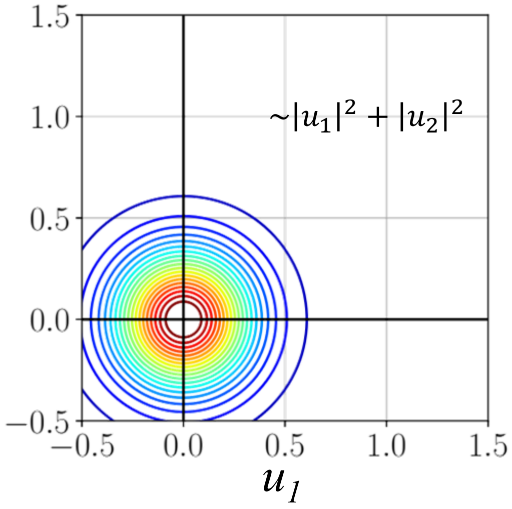
Student‘s t PDF

Sparse Bayesian Learning for modeling sources, cont'd

likelihood × prior ~ posterior

Gaussian Prior





highest probability:

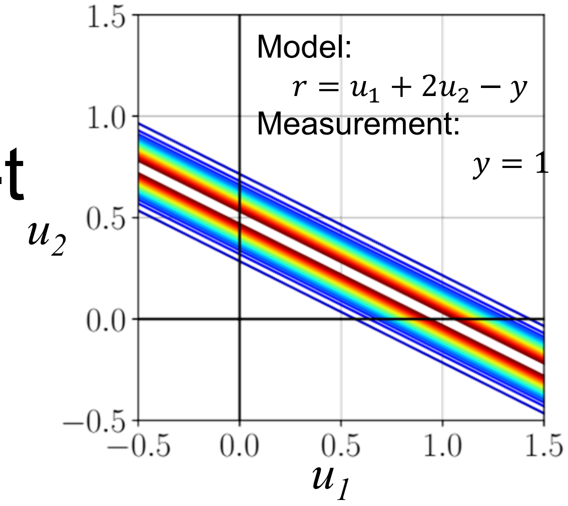
$u_1 = 0.2$

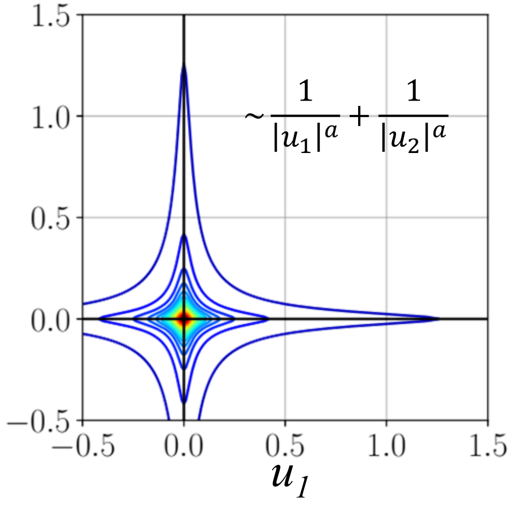
$u_2 = 0.3$

$\#\{u_i \neq 0\} = 2$

not sparse!

Student's-t Prior





highest probability:

$u_1 = 0$

$u_2 = 0.5$

$\#\{u_i \neq 0\} = 1$

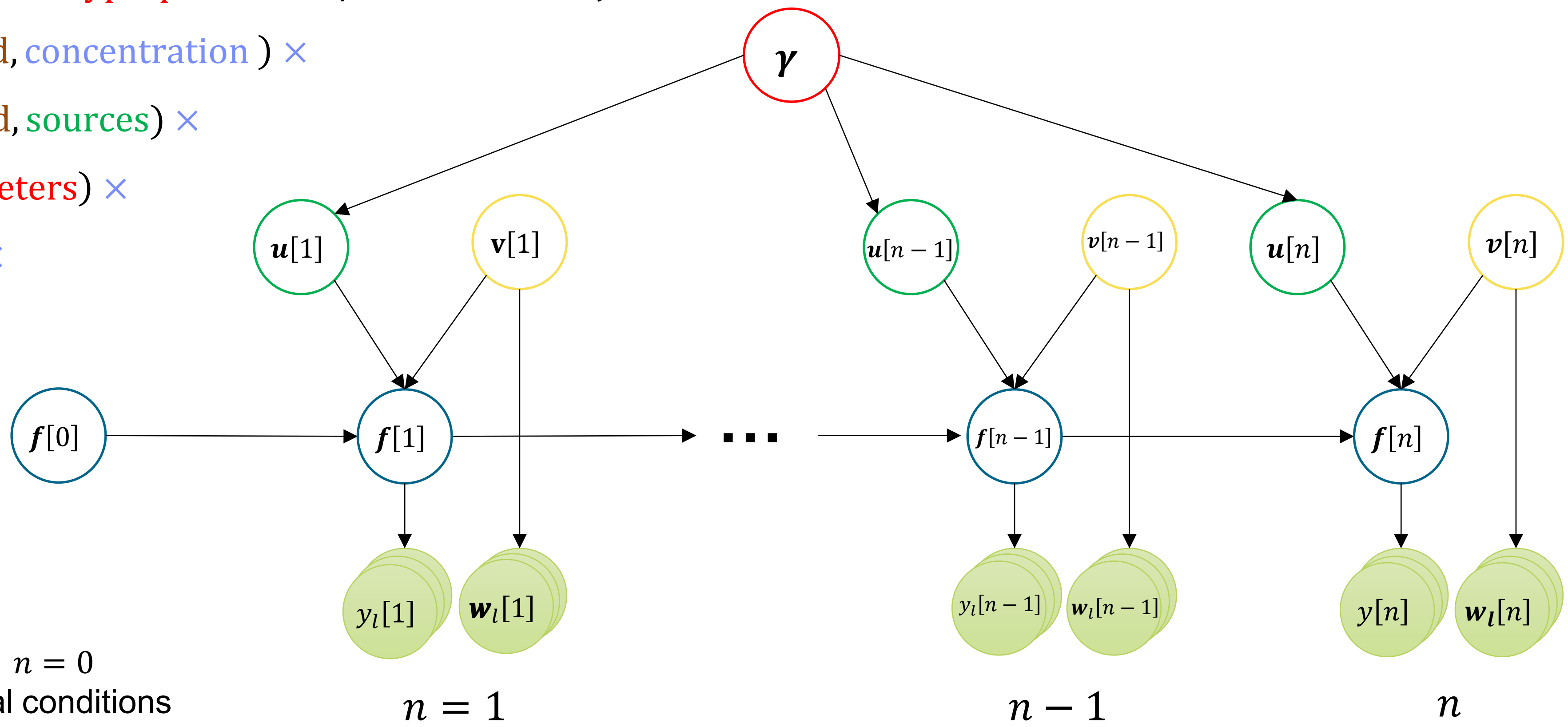
sparse

Probabilistic modeling of PDE

- Joint PDE can be represented with a classical Hidden Markov model

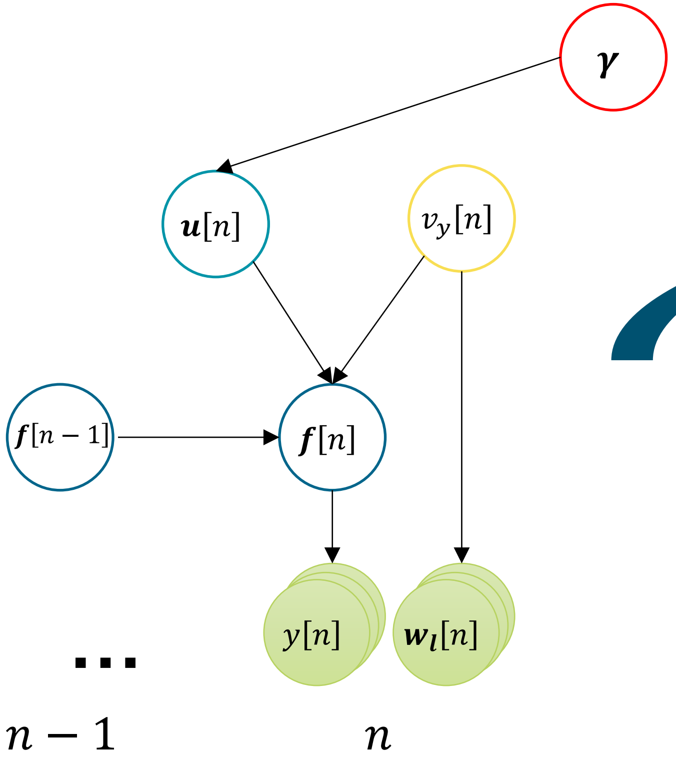
$p(\text{sources}, \text{concentration}, \text{wind}, \text{hyperparamters} \mid \text{measurements}) \propto$

model fit	$p(\text{measurement} \mid \text{wind}, \text{concentration}) \times$
model	$p(\text{concentration} \mid \text{wind}, \text{sources}) \times$
prior	$p(\text{source} \mid \text{hyperparameters}) \times$
hyperprior	$p(\text{hyperparameters}) \times$
prior	$p(\text{wind})$



Probabilistic Inference
From HMM to a factor graph

- Graphical model of the PDE

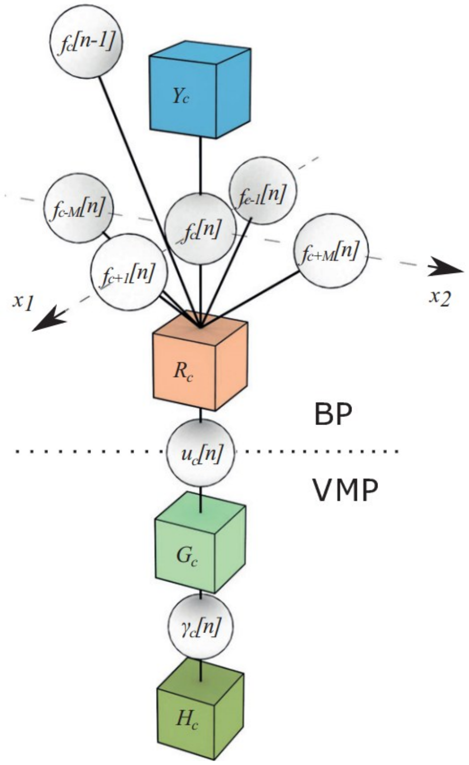


Joint PDF

$p(\text{sources}, \text{concentration}, \text{wind}, \text{hyperparamters} \mid \text{measurements}) \propto$

$$\prod_{\text{Samples}} \text{Fit} \prod_{\text{Cells}} \text{Model} \prod_{\text{Cells}} \text{Prior} \prod_{\text{Cells}} \text{Hyperprior}$$

- Factor graph representation of a single spatial cell



○ variable node = random variables

□ factor node = relation between variables

Probabilistic Inference on Factor Graphs
Message passing

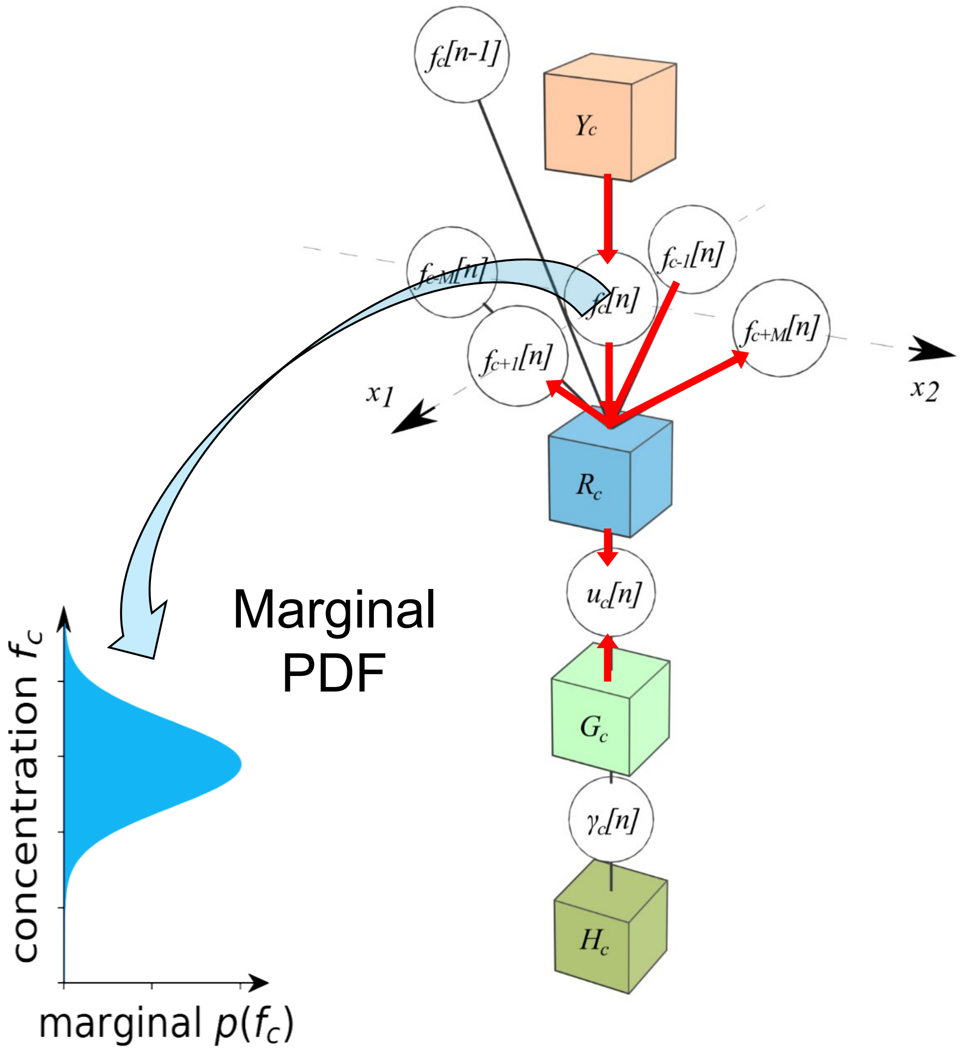
Message Passing Algorithm

(sum-product algorithm; loopy Belief Propagation)

Update Rules

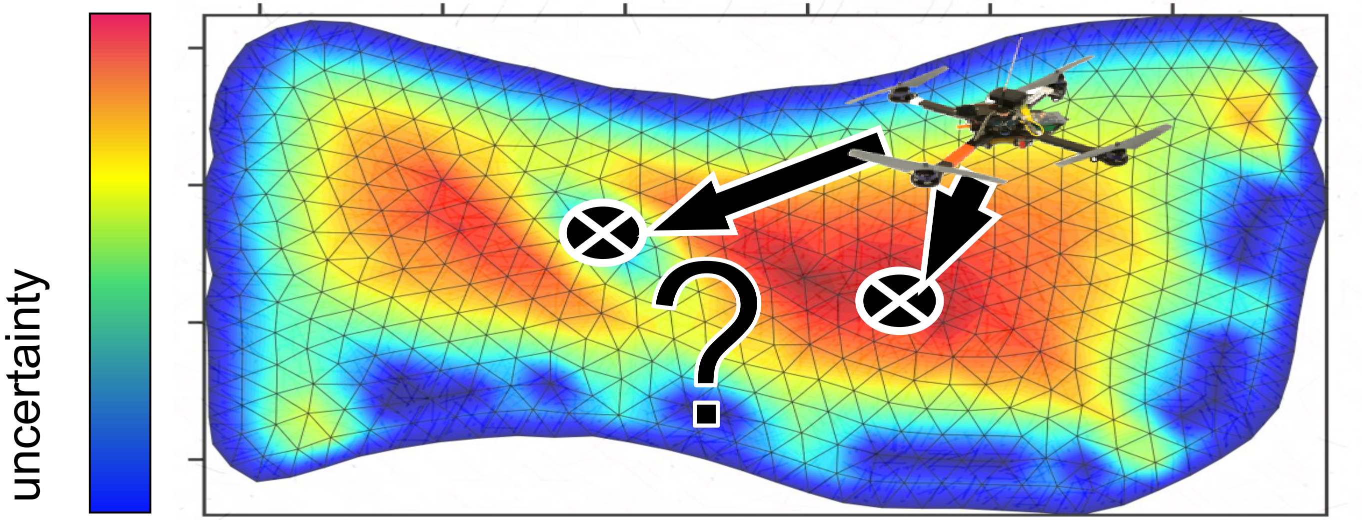
$$m_{\square_i \rightarrow \bigcirc_j} = \int \dots \int f(\bigcirc_1 \dots \bigcirc_M) \prod_{k \neq j}^{K} m_{\bigcirc_m \rightarrow \square_i} \, \mathrm{d}\bigcirc_1 \dots \mathrm{d}\bigcirc_M$$

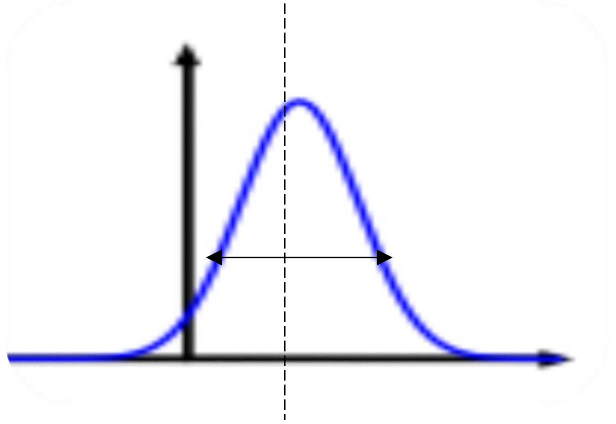
$$m_{\bigcirc_i \rightarrow \square_j} = \prod_{k \neq j}^{K} m_{\square_m \rightarrow \bigcirc_i}$$



How to utilize the learned model?
Exploration strategy

Uncertainty-driven exploration: optimal experiment design





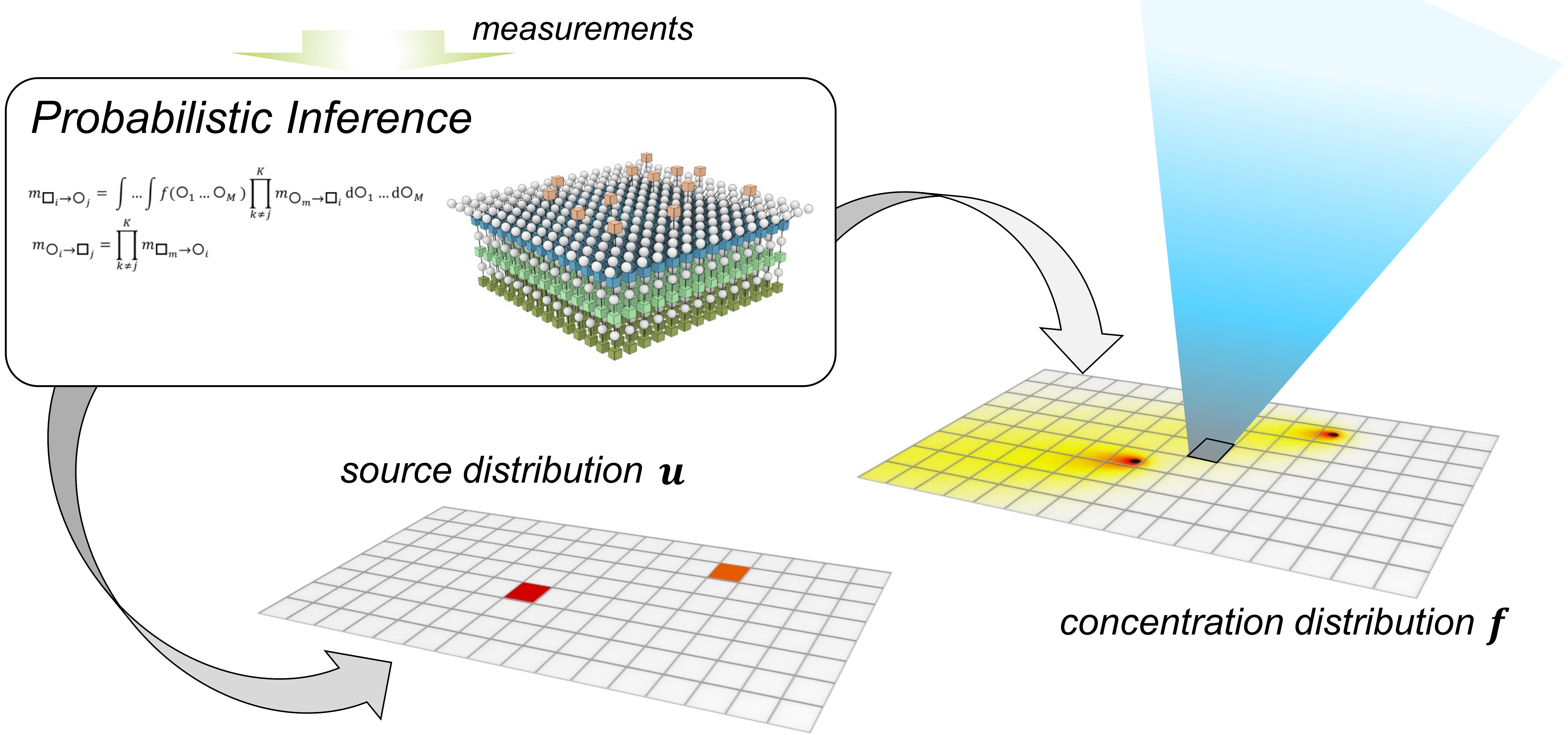
Uncertainty Map:

- based on variance of posterior marginal distribution
- spatial description of uncertainty
- highest uncertainties = proposals for multi agent system

uncertainty of parameter estimate:
→ measurements at locations with high uncertainty improve parameter estimation

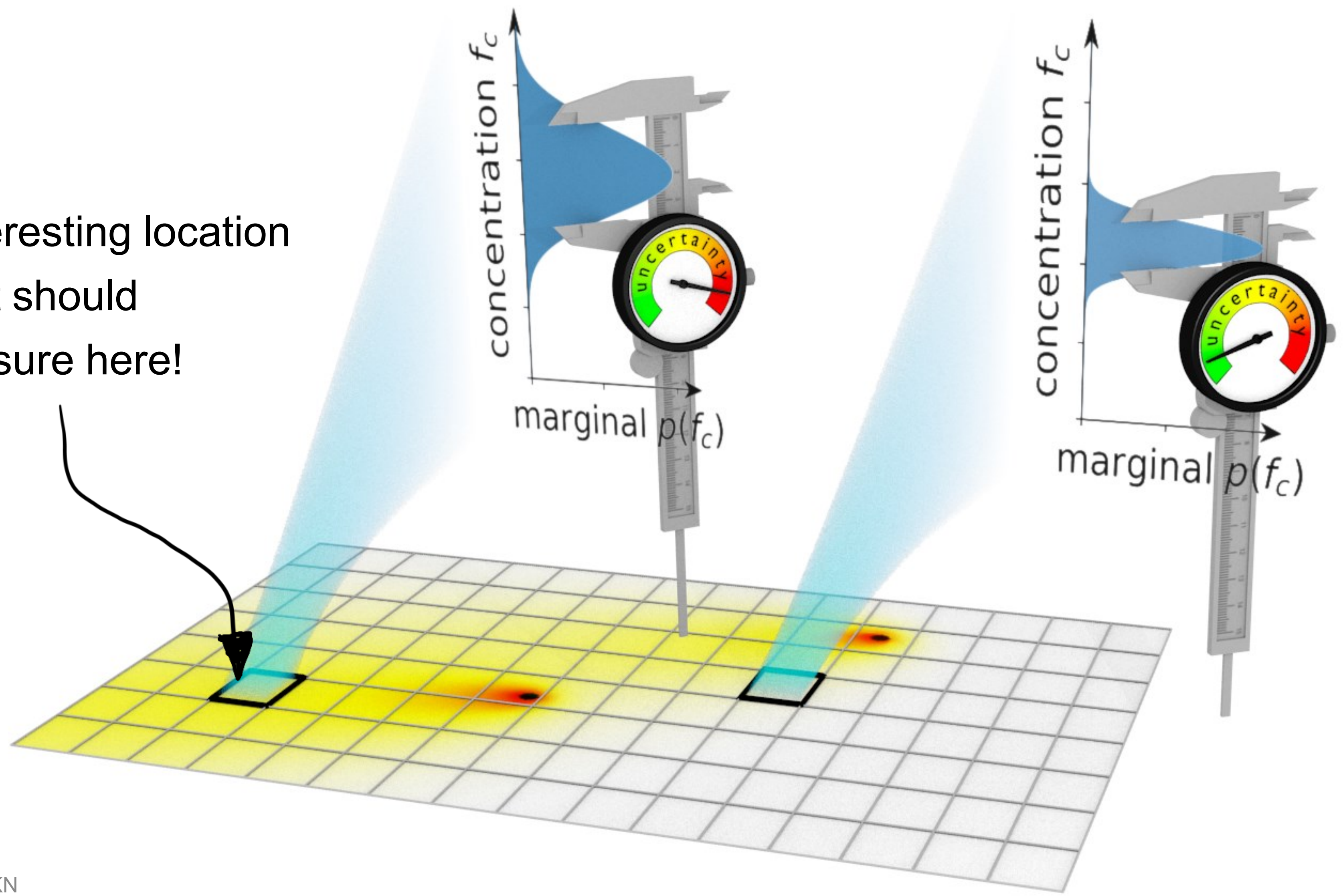
Cooperative Exploration Of Spatial Dynamic Proc

Exploration strategy



Uncertainty Driven Exploration Strategy

more interesting location
→ robot should
measure here!



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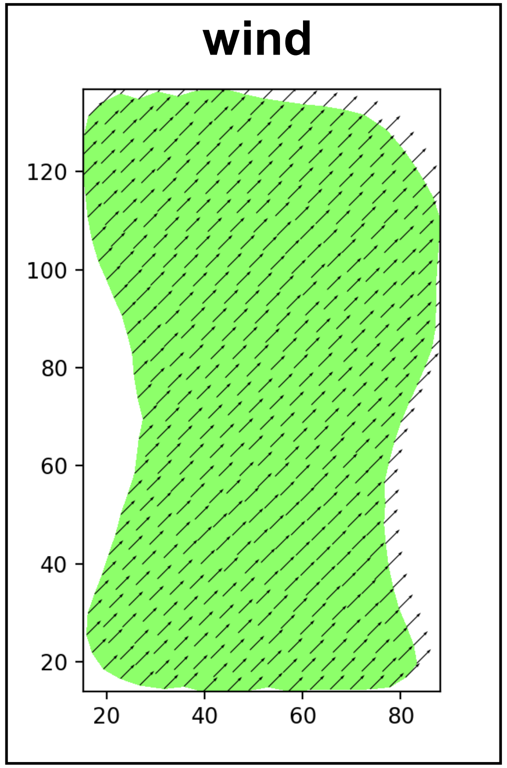
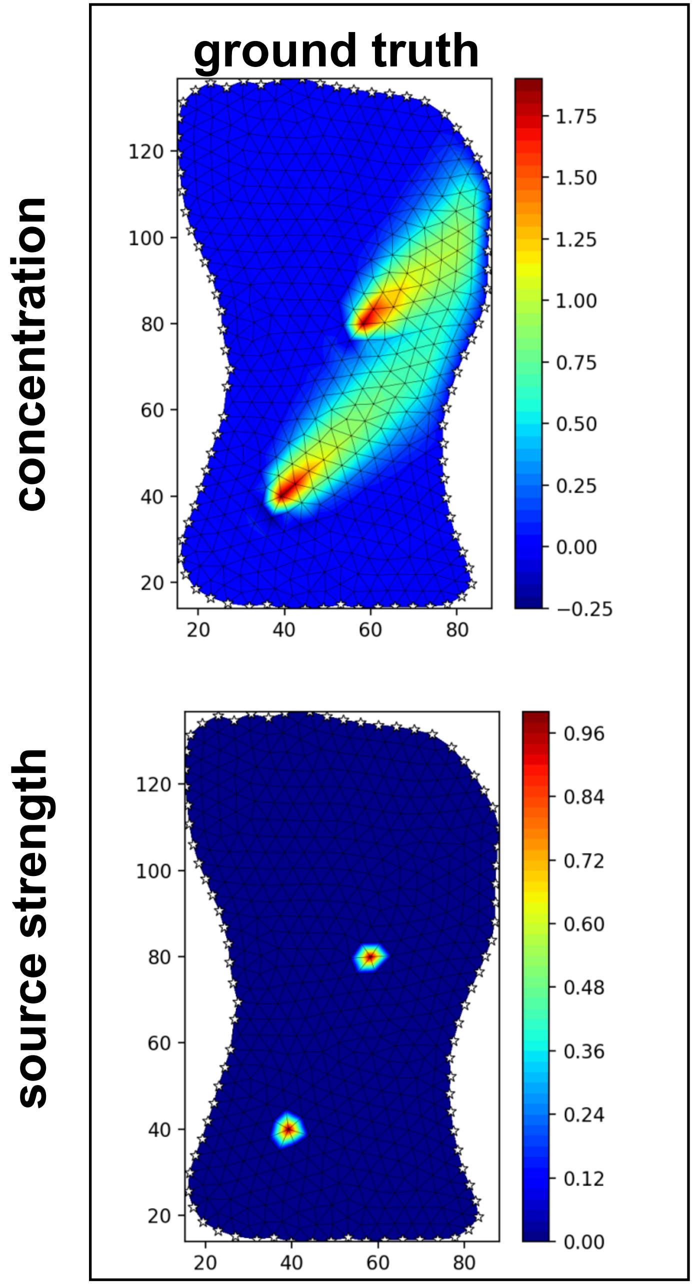
Simulation

250 measurements





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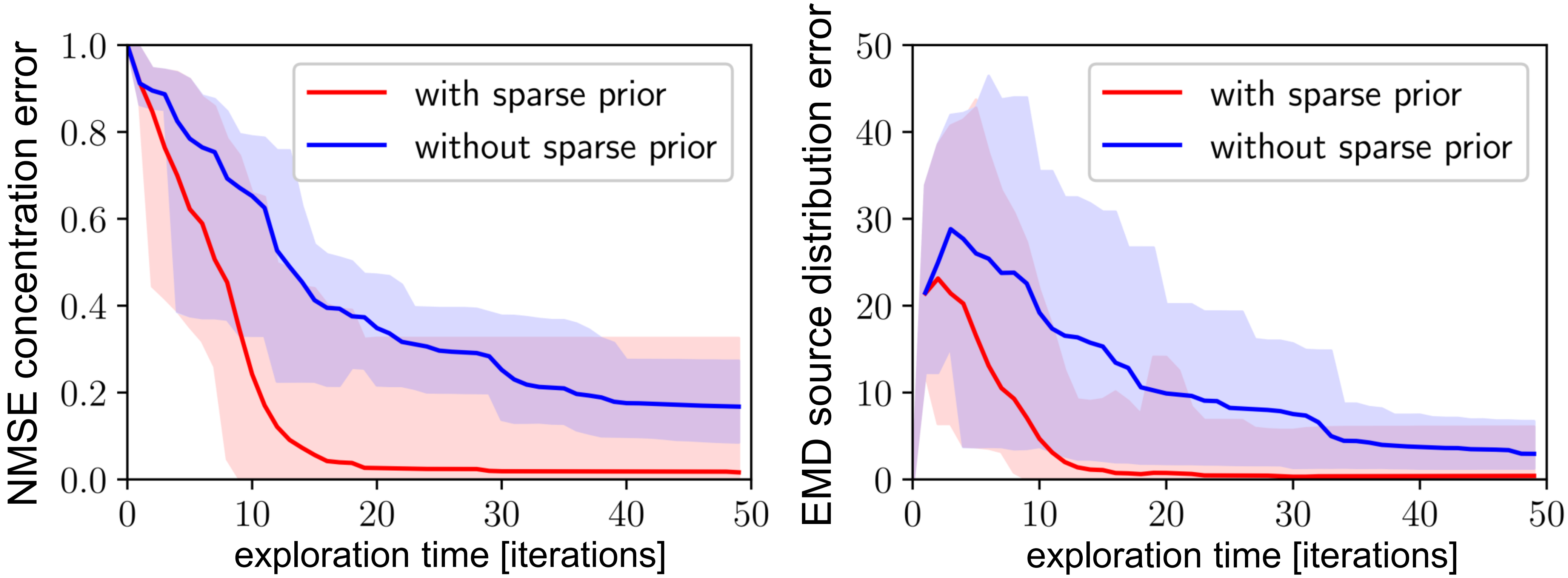
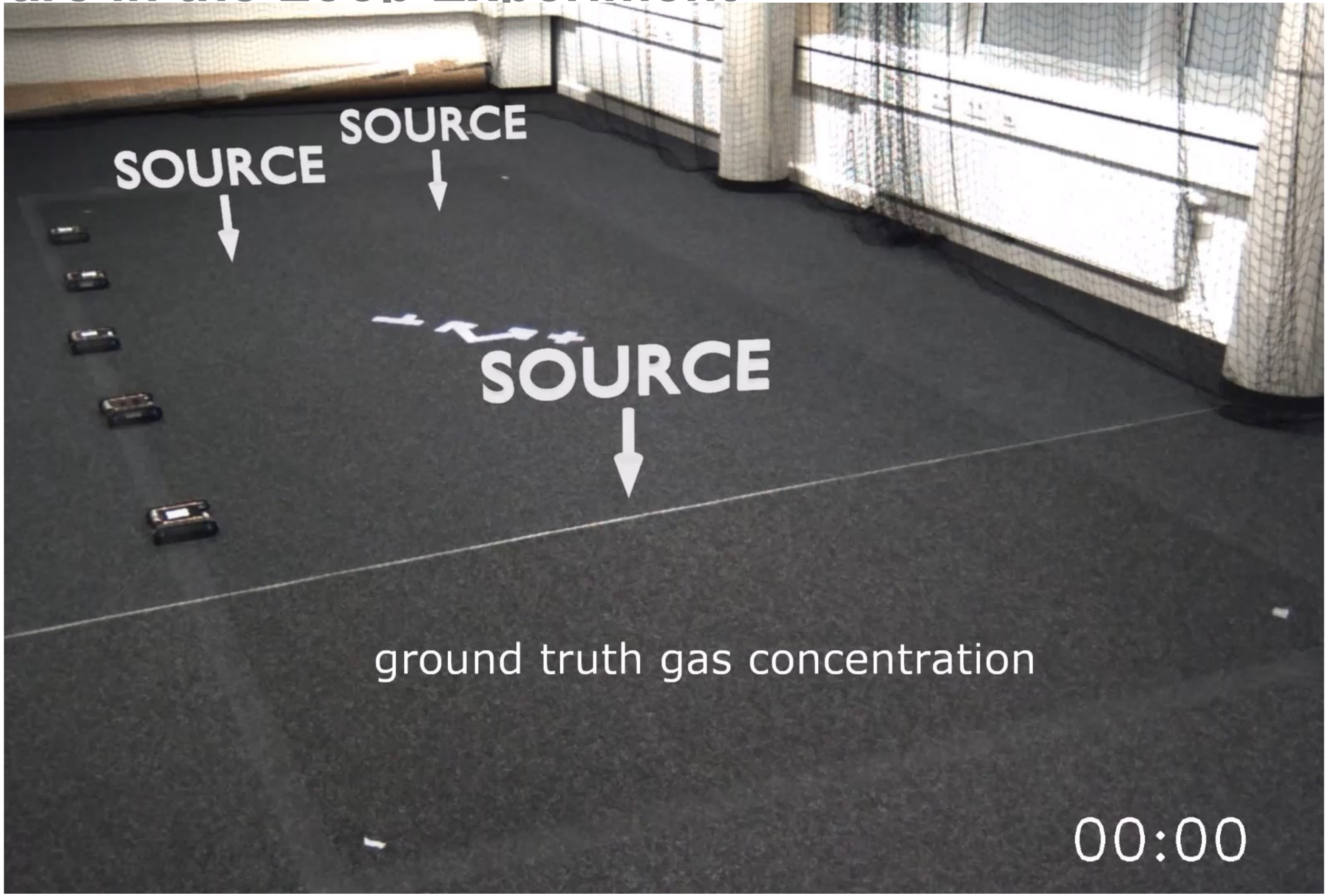
Impact of Sparsity Inducing Prior



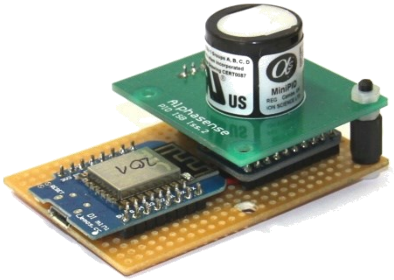
Simulation setup:

- up to 3 sources (random position)
- environment discretized by 676 points/cells
- 5 robots
- averaged over 45 simulation runs

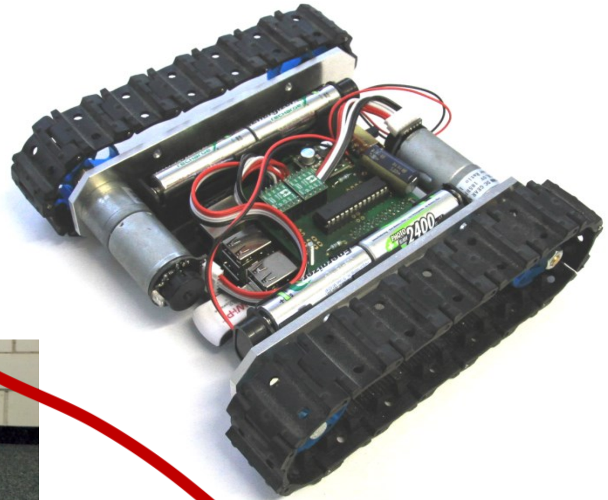
Hardware in the Loop Experiment

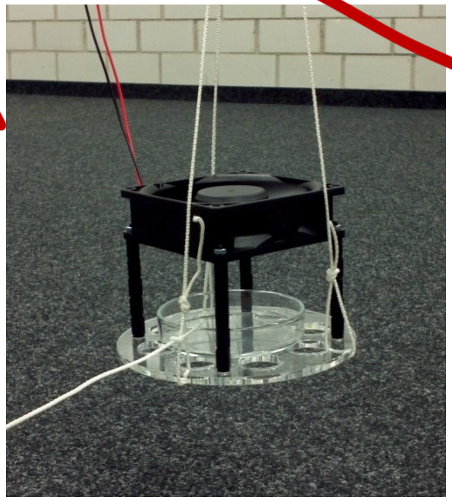


Real-World Experiment: Setup



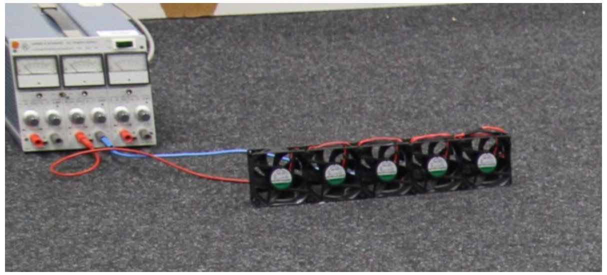
photoionization detector



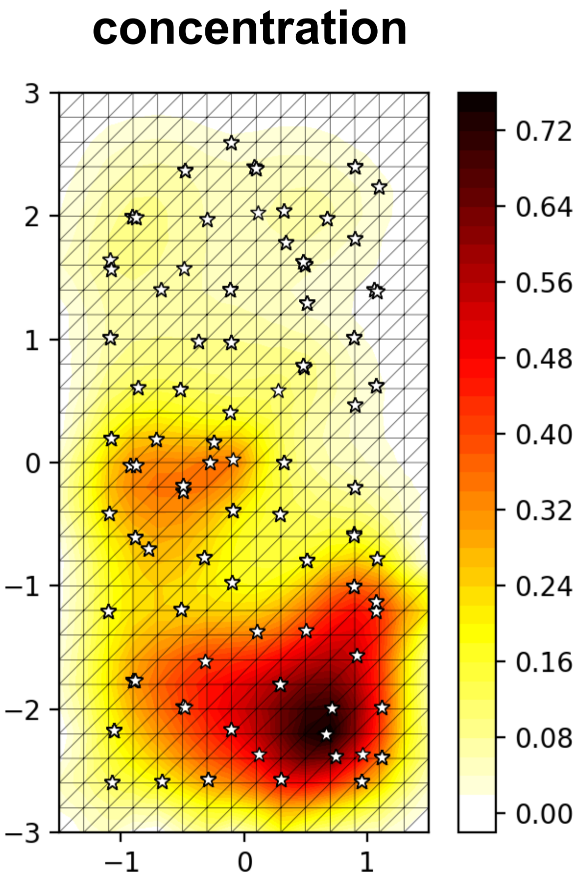


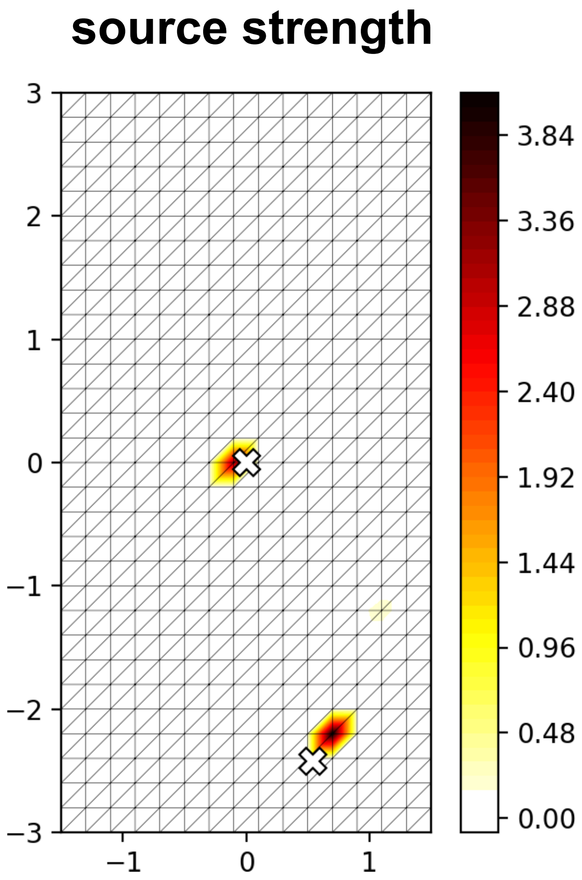
ethanol vapor source

artificial air flow (fan array)

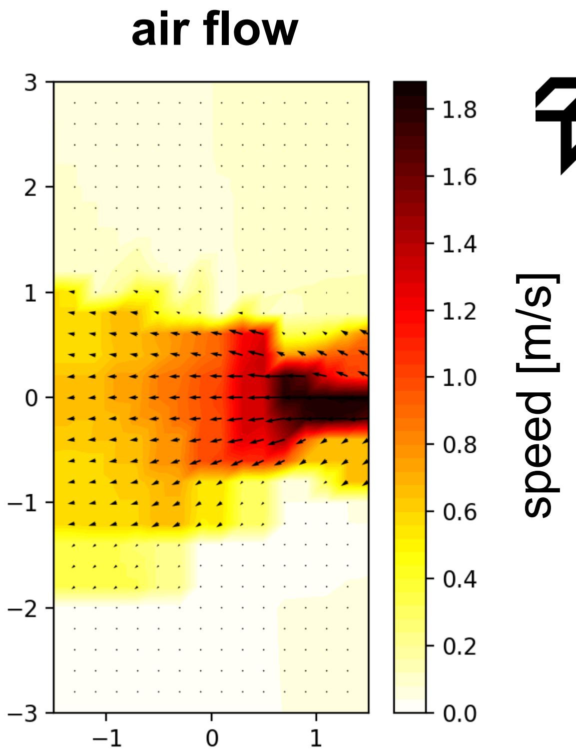


Real-World Experiment

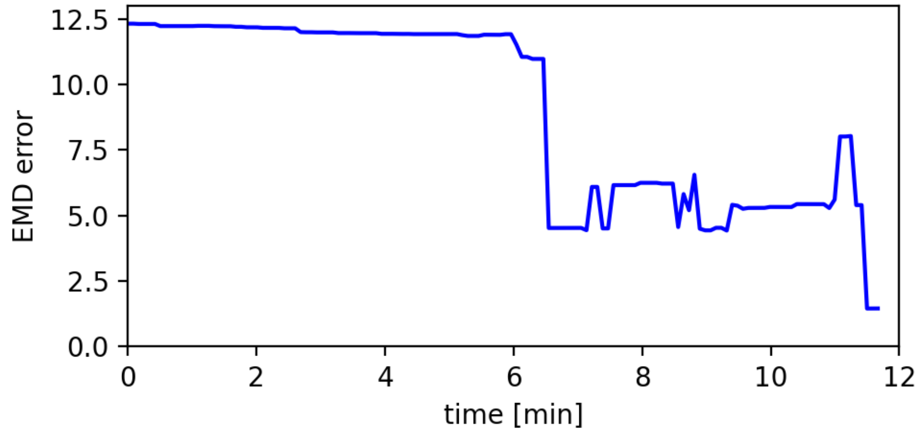




11 min



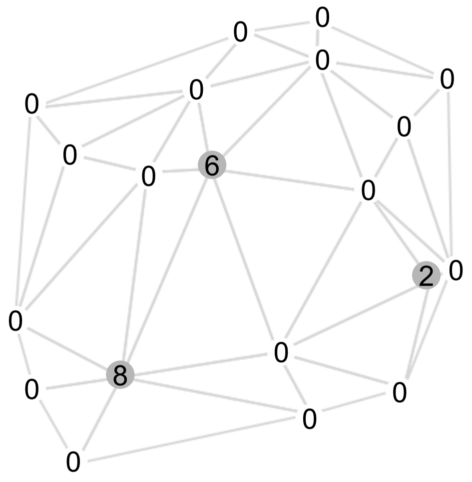
DLR



ANALYTICAL METHODS

Super-resolution Gas Source Localization
Beyond grid

- Gridded methods

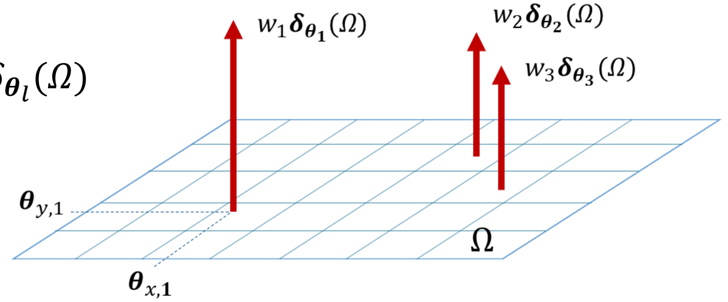


$$\mathbf{u} = \begin{bmatrix} 0 \\ 6 \\ 0 \\ \vdots \\ 0 \\ 2 \\ 0 \\ \vdots \\ 8 \\ 0 \\ 0 \end{bmatrix} \in \mathbb{R}^N$$

- Typical Numerical complexity $\sim\mathcal{O}(N^3)$
- Linear in $\mathbf{u} \in \mathbb{R}^N$

- Off-Grid methods

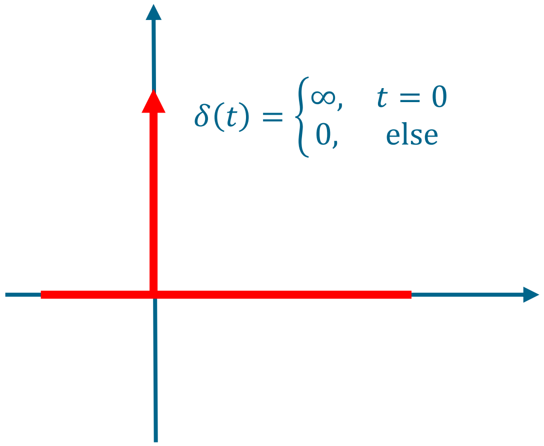
$$u(\boldsymbol{x}) = \sum_{l=1}^{L_{\text{true}}} w_l \, \delta_{\boldsymbol{\theta}_l}(\Omega)$$



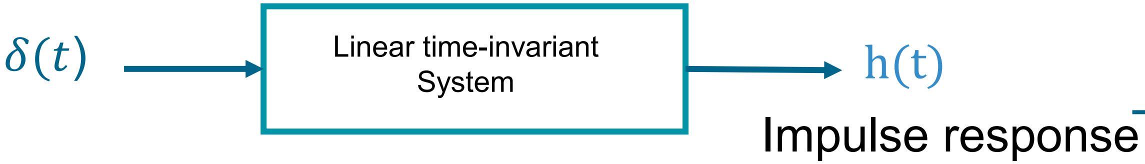
$$\boldsymbol{\Theta} = \begin{bmatrix} \boldsymbol{\theta}_1 \\ w_1 \\ \vdots \\ \boldsymbol{\theta}_{L_{\text{true}}} \\ w_{L_{\text{true}}} \end{bmatrix} \in \mathbb{R}^{3L_{\text{true}}}$$

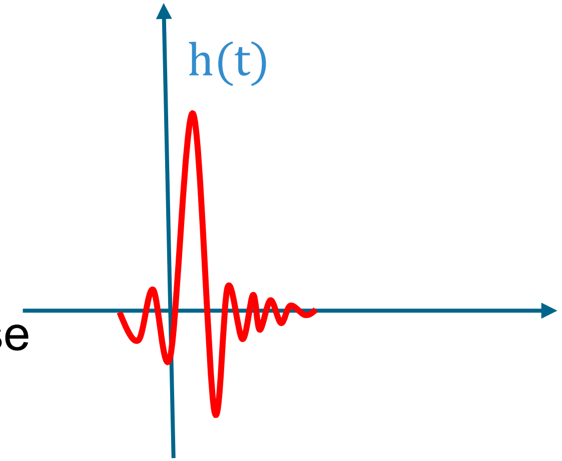
- Typical Numerical complexity $\sim\mathcal{O}\left({L_{\text{true}}}^3\right)$
- Nonlinear in $u(x)$ (or equivalently, in $\boldsymbol{\Theta}$)

Super-resolution Gas Source Localization
Green‘s function method

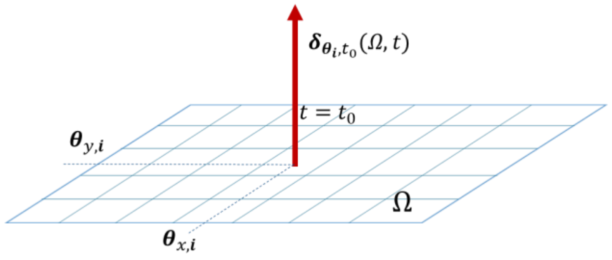


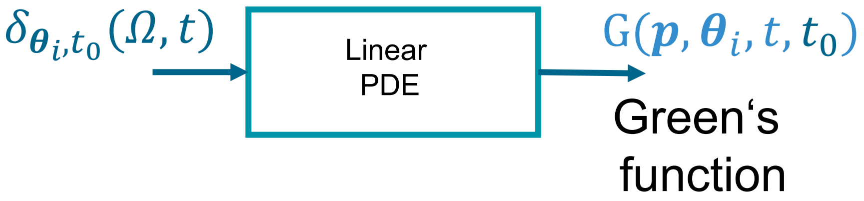
Linear systems and impulse response

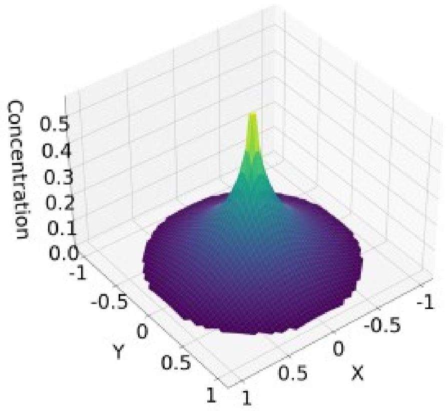




Green’s function method for solving PDE







For any number of sources:

$$f(\boldsymbol{p},t) = \sum_{i=1}^{L} w_i G(\boldsymbol{p},\boldsymbol{\theta}_i,t,t_i)$$

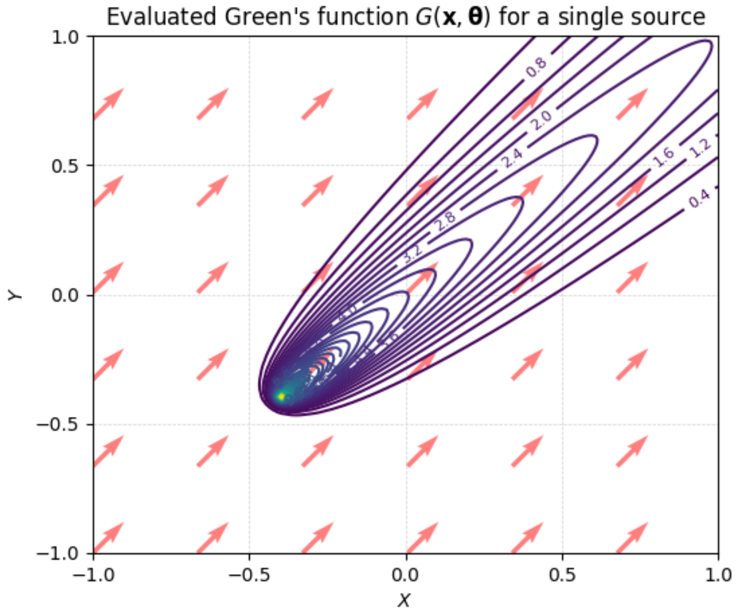
Green's function method for Advection-Diffusion

Simplification: Time-invariant advection-diffusion

Physics of smoke dispersion: Advection-Diffusion Equation

$$\boldsymbol{v}(\boldsymbol{p})^{\mathsf{T}}\nabla f(\boldsymbol{p}) - \kappa\Delta f(\boldsymbol{p}) = \sum_{i=1}^{L} w_i \delta_{\boldsymbol{\theta}_i}(\Omega), \quad \boldsymbol{p}, \boldsymbol{\theta}_i \in \Omega \subset \mathbb{R}^2$$

$$\text{s.t.} \quad f(\boldsymbol{p}) = 0, \ \boldsymbol{p} \in \partial\Omega$$



Green's function for unbounded domain

Analytic solution for $\Omega = [-\infty, \infty] \times [-\infty, \infty]$

$$G(\boldsymbol{p}, \boldsymbol{\theta}) = \frac{1}{2\pi\,\kappa}\, e^{\boldsymbol{v}(\boldsymbol{p})^T(\boldsymbol{p}-\boldsymbol{\theta})} K_0\left(\frac{\|\boldsymbol{p}-\boldsymbol{\theta}\|\,\|\boldsymbol{v}(\boldsymbol{p})\|}{2\,\kappa}\right)$$

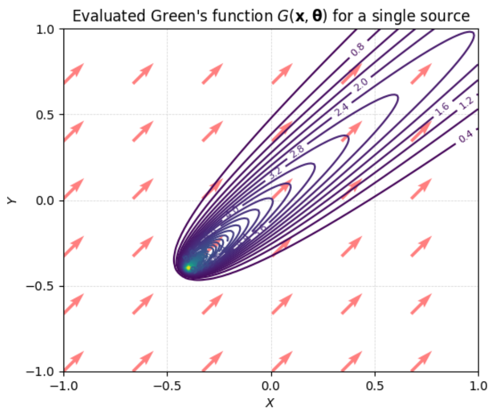
$K_0()$ is zero-order modified Bessel function

Signal model

Green's function for Poisson's equation on unit circle.

$$G(\boldsymbol{p},\boldsymbol{\theta}) = \frac{1}{2\pi\,\kappa}\, e^{\boldsymbol{v}(\boldsymbol{p})^T(\boldsymbol{p}-\boldsymbol{\theta})} K_0\left(\frac{\|\boldsymbol{p}-\boldsymbol{\theta}\|\,\|\boldsymbol{v}(\boldsymbol{p})\|}{2\,\kappa}\right)$$

$\boldsymbol{\theta}$ - Source location



Discretization of the PDE model:

- Discretize Ω into N of grid cells $C_i \in \Omega,\ i = 1 \ldots N$
- For each cell C_i with center $\boldsymbol{p}_i$ assume $f(\boldsymbol{p}) = \text{const}$
- Locations $\boldsymbol{\Theta}$ **are not** on the grid

Concentration

$$\boldsymbol{f} = \begin{bmatrix} f(\boldsymbol{p}_1) \\ f(\boldsymbol{p}_2) \\ f(\boldsymbol{p}_3) \\ \cdots \\ f(\boldsymbol{p}_N) \end{bmatrix}$$

Green's function

$$\boldsymbol{g}(\boldsymbol{\theta}_l) = \begin{bmatrix} G(\boldsymbol{p}_1, \boldsymbol{\theta}_l) \\ G(\boldsymbol{p}_2, \boldsymbol{\theta}_l) \\ G(\boldsymbol{p}_3, \boldsymbol{\theta}_l) \\ \cdots \\ G(\boldsymbol{p}_N, \boldsymbol{\theta}_l) \end{bmatrix}$$

Solution of PDE

$$\boldsymbol{f} = \sum_{l=1}^{l} w_l \boldsymbol{g}(\boldsymbol{\theta}_l) = \boldsymbol{G}(\boldsymbol{\Theta})\boldsymbol{w}$$

Observations:

- Assume M noisy samples z_m of $f(\boldsymbol{p})$ are available,
- collected at $\boldsymbol{p}_m \in \Omega$
- Assume $N \gg M$

Sensing matrix

$$\boldsymbol{M} = \begin{bmatrix} \underbrace{0,0,0,\ldots,0,1,0,\ldots,0}_{1 \text{ at 1st measurement location}} \\ \vdots \\ \underbrace{0,1,0,0,0,0,0,0,\ldots,0}_{1 \text{ at } M\text{th measurement location}} \end{bmatrix} \in \mathbb{R}^{M\times N}$$

Measurement model

$$\boldsymbol{z} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ \cdots \\ z_M \end{bmatrix} = \boldsymbol{M}\boldsymbol{f} + \boldsymbol{\xi}$$

Sparse Bayesian Learning for GSL
Inference model in a probabilistic context

SBL prior

$$p(w_l) = \int p(w_l|\gamma_l)p(\gamma_l)\mathrm{d}\gamma_l$$

$$p(w_l|\gamma_l)p(\gamma_l) \rightarrow N(w|0,\gamma_l) \times \text{const}$$

Full posterior

$$p(\boldsymbol{f},\boldsymbol{w},\boldsymbol{\gamma},\boldsymbol{\Theta}|\mathcal{Z}) \propto p(\mathcal{Z}|\boldsymbol{f})p(\boldsymbol{f}|\boldsymbol{w},\boldsymbol{\Theta})p(\boldsymbol{w},\boldsymbol{\gamma})p(\boldsymbol{\Theta})$$

- $p(\mathcal{Z}|\boldsymbol{f})$: **Likelihood / Measurements**
- $p(\boldsymbol{f}|\boldsymbol{w},\boldsymbol{\Theta})$: **PDE Model** $\boldsymbol{f} = \boldsymbol{G}(\boldsymbol{\Theta})\boldsymbol{w}$
- $p(\boldsymbol{w},\boldsymbol{\gamma})$: SBL prior
- $p(\boldsymbol{\Theta})$: **Const.**

Optimization strategy (nonlinear)

Initial source locations $\widehat{\boldsymbol{\theta}}$ →

Estimation of source support (SBL)
(fixed source locations $\widehat{\boldsymbol{\theta}}$)

$$\max_{\boldsymbol{\gamma},\boldsymbol{w}} \left\{ p(\boldsymbol{w},\boldsymbol{\gamma},\widehat{\boldsymbol{\Theta}}|\mathcal{Z}) = \int p(\boldsymbol{f},\boldsymbol{w},\boldsymbol{\gamma},\widehat{\boldsymbol{\Theta}}|\mathcal{Z})\mathrm{d}\boldsymbol{f} \right\}$$

Standard SBL for linear models

Source support estimate: $\widehat{\boldsymbol{\gamma}}, \widehat{\boldsymbol{w}}$ →

Estimation of source locations
(Fix support $\widehat{\boldsymbol{\gamma}}$)

$$\min_{\boldsymbol{\Theta},\boldsymbol{w}} \left\{ J(\boldsymbol{\Theta}) = -\log p(\mathcal{Z}|\boldsymbol{f}) \right\}, \text{ s.t. } \quad \boldsymbol{f} = \boldsymbol{G}(\boldsymbol{\Theta})\boldsymbol{w}$$

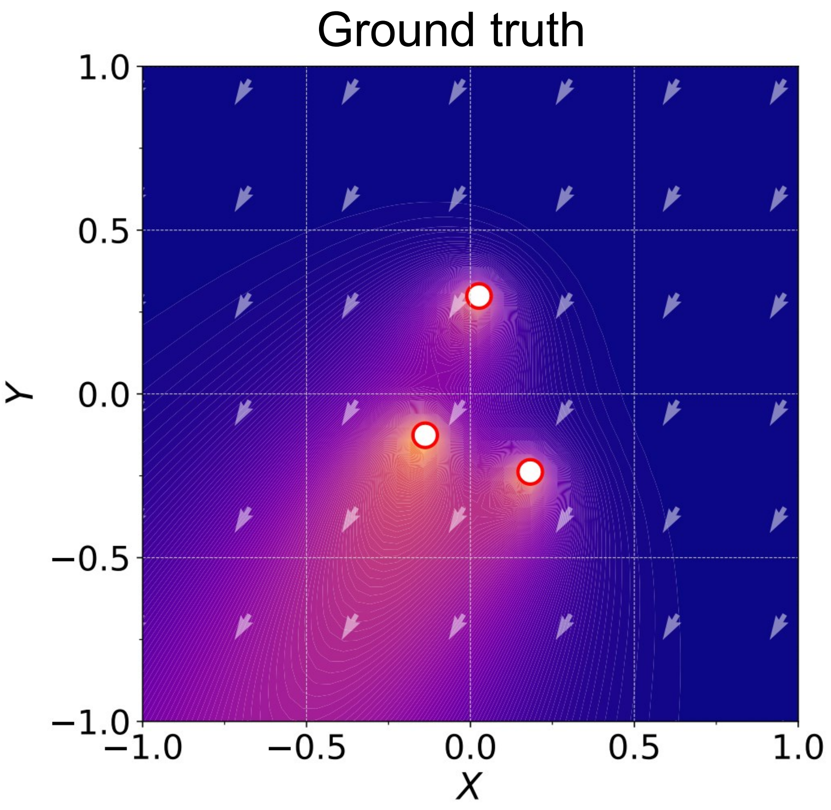
Nonlinear optimization

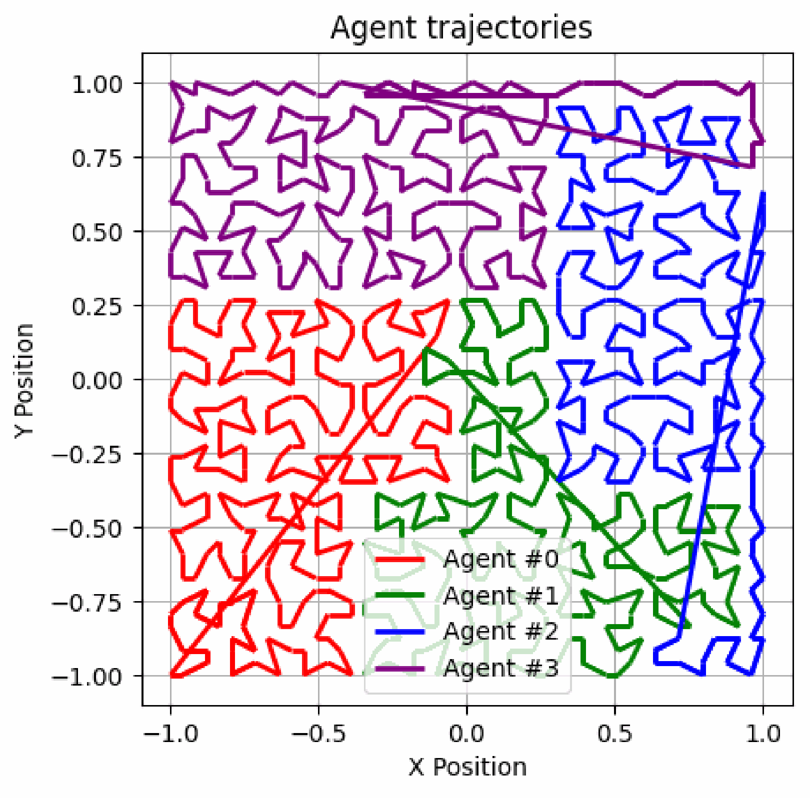
← Source location estimation $\widehat{\boldsymbol{\Theta}}, \widehat{\boldsymbol{w}}$

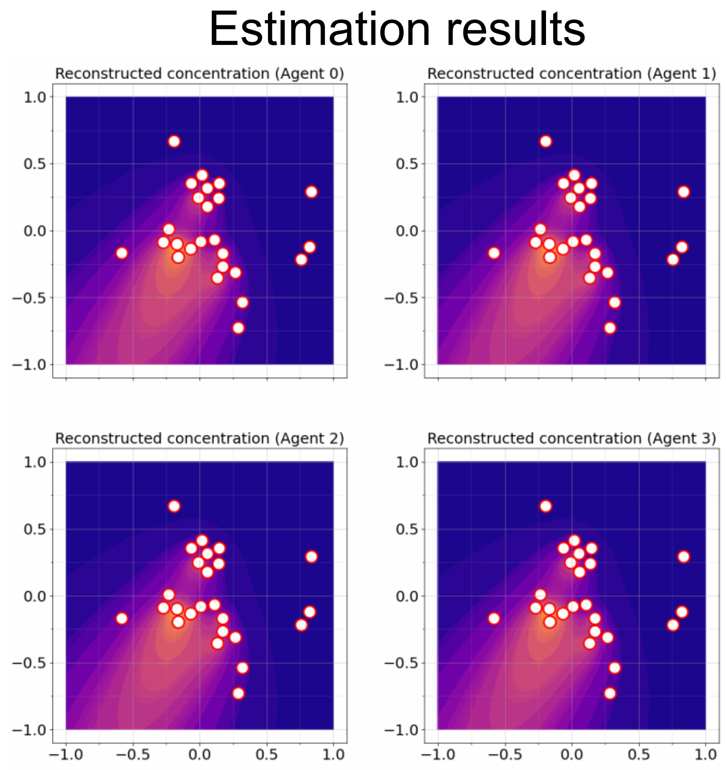
*D. Shutin, T. Wiedemann, and P. Hinsen, "Detection and estimation of gas sources with arbitrary locations based on Poisson's equation," *IEEE Open Journal of Signal Processing*, vol. 5, pp. 359–373, 2024.

SBL with Green‘s function method
Optimization over a network

- Swarm-based Sparse Bayesian Learning for smoke source localization







SBL based Cramer-Rao Lower Bound for source localization
Single source

Type II Likelihood function:

$$p(\boldsymbol{z}|\boldsymbol{\gamma},\boldsymbol{\theta}) \propto |\boldsymbol{\Sigma}_{\gamma}(\boldsymbol{\theta})|^{-\frac{1}{2}} \mathrm{e}^{-\frac{1}{2}\boldsymbol{z}^{\mathsf{T}}\boldsymbol{\Sigma}_{\gamma}(\boldsymbol{\theta})^{-1}\boldsymbol{z}}$$

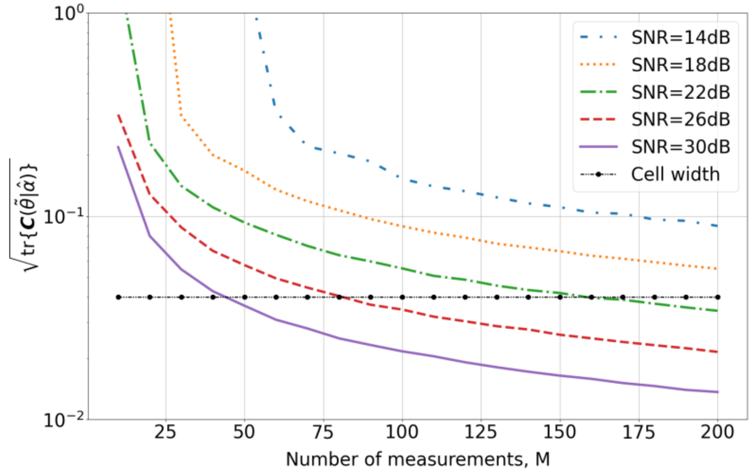
$$\boldsymbol{\Sigma}_{\gamma}(\boldsymbol{\theta}) = \lambda_{\xi}^{-1}\boldsymbol{I} + \gamma\ \boldsymbol{M}\boldsymbol{g}(\boldsymbol{\theta})\boldsymbol{g}(\boldsymbol{\theta})^{\mathsf{T}}\boldsymbol{M}^{\mathsf{T}}$$

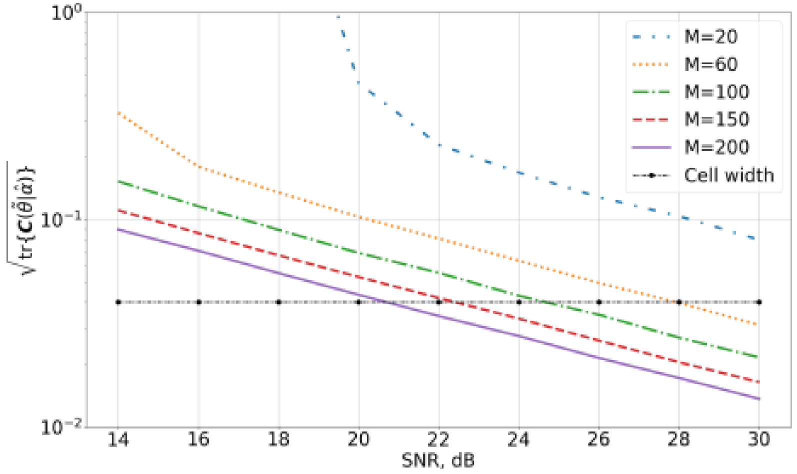
γ - Source sparsity parameter

$\boldsymbol{\theta}$ - Source location

Corresponding Fisher information

$$\boldsymbol{I}_{\eta,\epsilon}(\gamma,\boldsymbol{\theta}) = \mathrm{tr}\left\{\boldsymbol{\Sigma}_{\gamma}(\boldsymbol{\theta})^{-1}\frac{\partial\boldsymbol{\Sigma}_{\gamma}(\boldsymbol{\theta})}{\partial\eta}\boldsymbol{\Sigma}_{\gamma}(\boldsymbol{\theta})^{-1}\frac{\partial\boldsymbol{\Sigma}_{\gamma}(\boldsymbol{\theta})}{\partial\epsilon}\right\}$$





- **CRLB depends on Green's function and its derivative with respect to source location**
- **Key element for Uncertainty-driven exploration (for analytic methods)**

CONCLUSIONS AND FUTURE DEVELOPMENTS

Some concluding remarks

- Both numerical and analytical approaches introduce performance trade-offs
- Information (uncertainty) driven exploration can guide robots to better sampling locations
- … also both approaches can be implemented over a network of agents

- Green‘s functions can be approximated with NNs for arbitrary domains
  - Can be quite efficient in terms of complexity
  - Feed-forward approaches are useful, but need to cope with measures
  - Fourier Neural Operators can be better for PDEs than DNNs

- Wind plays a crucial role in the propagation modeling
  - Unknown wind can make the problem even more nonlinear
  - Leads to numerical CFD approaches…


Thank you
for your attention!